Symmetry and Conservation in Spacetime

Laura M. Becker

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1 Application of Group Theory in Physical Vector Spaces

The concept of groups originally came from several branches of mathematics, including geometry, number theory, and the theory of algebraic equations, and gradually led into the development of formal group theory beginning in the late 18th Century (O'Connor and Robertson, 1996). Since its formulation, group theory has been applied to almost every branch of physics. It is probably most well known for its usefulness in quantum mechanics, but group theory can also be used to find and explore symmetries of classical and relativistic physical systems. For example, the group of translations in space is a symmetry group for classical space because of the homogeneity of space. The conservation of linear momentum can be derived from this symmetry. The group of translations in time is also a symmetry group for classical time, because of the homogeneity of time. This symmetry can be used to derive the conservation of energy. The group of rotations of space, which is a symmetry group of space because of the isotropy of space, can be used to derive the conservation of angular momentum. Additionally, the group of Lorentz transformations is a symmetry group for spacetime which is used in special relativity.

1.1 Basic Group Theory

A *group* is a set G with an operation $*$ which associates any given ordered pair of elements $a, b \in G$ with a product $a * b$ which is also an element of G, such that the following conditions are satisfied:

1. The operation $*$ is associative (i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in G$);
2. There is an element $e \in G$ with the property that $a * e = a$ for all $a \in G$;
3. For each $a \in G$, there is an element $a^{-1} \in G$ with $a * a^{-1} = e$.

The element e is called the *identity* for the group, and the operation $*$ is referred to as the *group multiplication* for the group [6, p. 12]. $a * b$ is often denoted simply as ab. A single element of a group can be raised to a positive integral power by the following definition: $a^1 = a$; if $n > 1$, $a^n = a^{n-1} * a$. $<a>$ is defined as the set of all powers of the element a which exist in the given group [7, p. 3].

There are two major ways in which groups can be related to each other. The first is by a *homomorphism*, which is a unidirectional mapping from one group to another. A homomorphism is defined as follows: Let G and H be groups; a homomorphism $f : G \to H$ is a unidirectional mapping which preserves group multiplication. That is, if $g_i \in G$ implies $h_i \in H$ and $g_1 * g_2 = g_3$, then $h_1 * h_2 = h_3$ [7, p. 6]. There is a homomorphism from the group of permutations of three objects, called S_3, to the group consisting of the numbers 1 and -1 under normal multiplication. This homomorphism is defined as follows:

$$\begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix} \to 1, \quad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix} \to 1, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \to 1, \tag{1}$$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \to -1, \quad \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \to -1, \quad \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix} \to -1,$$

where the permutation

$$\begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$$

sends the object in position a to position a', the object in position b to position b', and the object in position c to position c' [8, p. 7]. We see that the *cyclic* permutations, those which shift the objects by a fixed number of places, with the elements shifted off the end inserted back at the beginning, go to 1 under the homomorphism, while those permutations which shift different objects by different numbers of places go to -1 under the homomorphism [9].

The other way in which groups can be related is by an *isomorphism*, which is a bidirectional mapping between two groups. An isomorphism is defined as follows: Let G and H be groups; an isomorphism is a mapping which preserves group multiplication and which is also a one-to-one correspondence. That is, if $g_i \in G$ is equivalent to $h_i \in H$ and $g_1 * g_2 = g_3$, then $h_1 * h_2 = h_3$ [6, p. 17]. Two groups G and H are said to be *isomorphic*, denoted by $G \simeq H$, if there is an isomorphism $f : G \rightarrow H$ [7, p. 6]. There is an isomorphism from the group of permutations of two objects, called S_2, to the group consisting of the numbers 1 and -1 under normal multiplication. This isomorphism is defined as follows:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \rightarrow 1, \quad \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \rightarrow -1. \tag{2}$$

The group consisting of space inversion (the transformation $\mathbf{x} \rightarrow -\mathbf{x}$ of 3-dimensional space) and the identity transformation ($\mathbf{x} \rightarrow \mathbf{x}$) is also isomorphic to S_2 and to the group consisting of the numbers 1 and -1. Thus these three groups are all isomorphic to each other. An isomorphism is a special kind of relation called an *equivalence relation*. This means that the following three properties are true for all groups [6, p. 19]:

1. Each group is isomorphic to itself (i.e., $G \simeq G$ for any group G);
2. If $G \simeq H$, then $H \simeq G$;
3. If $G \simeq H$ and $H \simeq J$, then $G \simeq J$.

1.2 Classifying Groups and their Elements

Groups can be classified in a variety of ways. A given group is either *abelian* or *non-abelian*. A group G is said to be abelian if its group multiplication is commutative (i.e., $a * b = b * a$ for all $a, b \in G$). The set $\{a, b, e\}$, with group multiplication defined as follows: $ab = ba = e$, $aa = b$, $bb = a$, forms the group C_3. This group is a simple example of an abelian group [6, p. 13]. The group of symmetry transformations of an equilateral triangle, called D_3, is a non-abelian group. The transformations that make up this group are: the identity transformation (which leaves the triangle unchanged), reflection in the angle bisector of any of the three vertices, and rotation about the center by $\frac{2\pi}{3}$ or $\frac{4\pi}{3}$. The group multiplication for this group is defined as follows: ab denotes application of a followed by application of b. For a given reflection a and a given rotation b, ab is equivalent to another reflection c, but ba is equivalent to a different reflection d, where $d \neq c$. Thus the group of symmetry transformations of an equilateral triangle is a non-abelian group [6, p. 14].

A given group is also either *cyclic* or *non-cyclic*. A group G is cyclic if $G = <a>$ for some $a \in G$ [7, p. 12]. The element a is known as the *generator* of the group G. The group C_3, discussed above as an example of an abelian group, is cyclic. Since $aa = b$, $b = a^2$, and since $ab = e$, $a * a^2 = e$, so $a^3 = e$. Thus the elements of the group are a, $b = a^2$, and $e = a^3$, which form $<a>$ for the group. The cyclic groups are denoted by C_n, where n is any positive integer. The group C_n has the following form: $\{a, a^2, ..., a^{n-1}, a^n = e\}$ [6, p. 13]. The group of symmetry transformations of a rectangle, called D_2, is an example of a non-cyclic group. It consists of the identity transformation (which leaves the rectangle unchanged), reflection about the vertical axis, reflection about the horizontal axis, and rotation about the center by π [6, p. 13,14]. This group is not cyclic because there is no one transformation in the group for which each other transformation is equivalent to some number of successive applications of that transformation. That is, there does not exist an element $a \in D_2$ such that for all elements $b \in D_2$ $b = a^n$, where n is some positive integer.

Another way to classify a group is by its *order*. The order of a group G, denoted by $|G|$, is the number of elements in G (if G is finite). The order of a cyclic group C_n is n. The order of the group D_2 is 4, and the order of the group D_3, discussed above as an example of a non-abelian group, is 6 [6, p. 13]. Because an isomorphism is a one-to-one correspondence, isomorphic groups must be of the same order.

The elements of a group can be classified within the group by the *classes* to which they belong. A class consists of all the elements of a group which are *conjugate* to each other. Two elements a, b in a group G are conjugate if there is another element p in G for which $b = pap^{-1}$. If a and b are conjugate elements of a group, we write $a \sim b$. Conjugation, like isomorphism, is an equivalence relation. Thus it has the following three properties [6, p. 19]:

1. Each element of a group G is conjugate to itself (i.e., $a \sim a$ for any element $a \in G$);

2. If $a \sim b$, then $b \sim a$;

3. If $a \sim b$ and $b \sim c$, then $a \sim c$.

The first property is true because, if $a \neq e$, then $a = eae^{-1} = eae = a$, and if $a = e$, then $e = pep^{-1} = pp^{-1} = e$ for any other element $p \in G$.

1.3 Representations of Groups on Vector Spaces

Groups are valuable tools in geometry and physics because they can be used to represent linear transformations on vector spaces. The spaces we refer to when thinking about physical systems, such as 2-dimensional Euclidean space ($\mathbf{R}^2$), 3-dimensional Euclidean space ($\mathbf{R}^3$), and spacetime ($\mathbf{R}^4$), are vector spaces. Often the set of solutions to an equation forms a vector space. There are also other types of vector spaces, such as those for which the "vectors" are actually functions. A set V is called a *vector space* if it is *closed* under addition and scalar multiplication and satisfies the vector space axioms. A set is said to be closed under the operations of addition and scalar multiplication if it has the following two properties:

1. $\alpha\mathbf{x} \in V$ for any $\mathbf{x}\in V$ and any scalar α;
2. $\mathbf{x} + \mathbf{y} \in V$ for any $\mathbf{x}, \mathbf{y} \in V$.

The vector space axioms are [10, p. 129]:

1. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$ for any $\mathbf{x}, \mathbf{y} \in V$;
2. $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z})$ for any $\mathbf{x}, \mathbf{y}, \mathbf{z} \in V$;
3. There exists an element $\mathbf{0}$ in V such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for each $\mathbf{x} \in V$;
4. For each $\mathbf{x} \in V$, there exists an element **-x** in V such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$;
5. $\alpha(\mathbf{x} + \mathbf{y}) = \alpha\mathbf{x} + \alpha\mathbf{y}$ for any scalar α and any $\mathbf{x}, \mathbf{y} \in V$;
6. $(\alpha + \beta)\mathbf{x} = \alpha\mathbf{x} + \beta\mathbf{x}$ for any scalars α, β and any $\mathbf{x} \in V$;
7. $(\alpha\beta)\mathbf{x} = \alpha(\beta\mathbf{x})$ for any scalars α, β and any $\mathbf{x} \in V$;
8. $1\mathbf{x} = \mathbf{x}$ for any $\mathbf{x} \in V$.

A linear transformation is a mapping $L : V \to W$, where V and W are vector spaces, such that [10, p. 188]

$$L(\alpha\mathbf{v}_1 + \beta\mathbf{v}_2) = \alpha L(\mathbf{v}_1) + \beta L(\mathbf{v}_2). \tag{3}$$

Since the multiplication of linear transformations on a linear vector space is associative, a set of invertible linear transformations which is closed with respect to multiplication of these transformations forms a group, called a *group of operators*, where *operator* is just another term for transformation. A *representation* of a group G is defined as a homomorphism $U : G \rightarrow U(G)$, where $U(G)$ is a group of operators on a linear vector space V. Representations which are isomorphisms are said to be *faithful*, whereas those which are not isomorphisms are said to be *degenerate* [6, p. 27].

1.4 The Groups of Translations

The group T_1 is the group of translations in one-dimensional Euclidean space. A translation by a distance x is denoted by $\mathrm{T}(x)$. The "state" of a particle at position x_0 is represented as $|x_0\rangle$. The operation of $\mathrm{T}(x)$ on $|x_0\rangle$ is:

$$T(x)|x_0\rangle = |x_0 + x\rangle \tag{4}$$

Elements of the group T_1 have the following properties [6, p. 89]:

1. $\mathrm{T}(x_1)\mathrm{T}(x_2)=\mathrm{T}(x_1 + x_2)$;
2. $\mathrm{T}(0)=E$, the identity transformation;
3. $\mathrm{T}(x)^{-1}=\mathrm{T}(-x)$.

The group T_n is the more general group of translations in n-dimensional Euclidean space. In this group, a translation is denoted by an n-dimensional vector rather than by a scalar distance, because in the space $\mathbf{R}^n$ a translation is a shifting of the space by a specific distance in a specific direction, so a vector is needed to characterize the translation. Thus a translation in $\mathbf{R}^n$ by a vector $\mathbf{x}$ is denoted by $\mathrm{T}(\mathbf{x})$. Then the operation of $\mathrm{T}(\mathbf{x})$ on $|\mathbf{x}_0\rangle$ is:

$$T(\mathbf{x})|\mathbf{x}_0\rangle = |\mathbf{x}_0 + \mathbf{x}\rangle \tag{5}$$

The elements of T_n have the corresponding properties to those listed above for T_1.

1.5 The Group of 3-dimensional Rotations

The group SO(3) is the group of rotations of 3-dimensional Euclidean space. The transformations in this group form a subset of the set of all transformations (linear and non-linear) of 3-dimensional Euclidean space. These transformations are represented by the set of real 3×3 matrices. The matrices in this set which are orthogonal form the set of linear transformations of 3-dimensional space which preserve the lengths of vectors. (An $n \times n$ matrix is orthogonal if $AA^T = I_n$, where I_n is the $n \times n$ identity matrix.) The determinant of an orthogonal matrix is either 1 or -1. The 3×3 orthogonal matrices with determinant 1 are the *pure rotations* of 3-dimensional space, and those with determinant -1 are the *rotation-reflections* of 3-dimensional space. (A rotation-reflection is a pure rotation combined with a reflection.) The group SO(3) is the group of pure rotations of 3-dimensional Euclidean space. Thus each element R of this group is represented by an orthogonal 3×3 matrix with determinant 1 [11, p. 1].

Each element R of SO(3) is uniquely determined by three continuous parameters. One of the most common parametrizations of the elements of SO(3) is the angle-and-axis parametrization. A given rotation R can be specified as $\mathrm{R}_{\hat{n}}(\psi)$, where $\hat{\mathbf{n}}$ is a unit vector pointing in the direction of the axis of rotation, and ψ is the angle of counterclockwise rotation about that axis. $\hat{\mathbf{n}}$ can be expressed in spherical coordinates as (r, θ, ϕ), and since the r-coordinate for any $\hat{\mathbf{n}}$ is 1, the angles θ and ϕ sufficiently specify $\hat{\mathbf{n}}$. Thus a rotation R can be parametrized by the three angles (ψ, θ, ϕ), where $0 \leq \psi \leq \pi$, $0 \leq \theta \leq \pi$, $0 \leq \phi < 2\pi$. Any rotation of 3-dimensional space can parametrized this way, since a rotation $\mathrm{R}_{\hat{n}}(\psi)$ where $\psi > \pi$ can be expressed as $\mathrm{R}_{-\hat{n}}(2\pi - \psi)$. We can associate each rotation R with a 3-dimensional vector $\mathbf{c}=\psi\hat{\mathbf{n}}$. Then the ends of these vectors constitute a solid sphere of radius π. However, there is not a one-to-one correspondence between the points in the sphere and the rotations in SO(3). This is because in this parametrization $\mathrm{R}_{-\hat{n}}(\pi)=\mathrm{R}_{\hat{n}}(\pi)$, which means that the two points at opposite ends of a diameter of the sphere are equivalent; that is, they are actually the same point. Such a sphere is *doubly connected*, meaning that there are two ways for a curve on the surface of the sphere to be closed, i.e., there are two classes of closed curves on the sphere. Curves in the first class can be continuously deformed into a point. These curves actually *look like* closed curves. Curves in the second class do not *look like* closed curves, but they have one endpoint at a point P on the surface of the sphere and the other endpoint also on the surface, at the opposite end of the diameter through P. Such a curve is closed because its endpoints are actually the same point P (since they are at opposite ends of the same diameter).

An interesting feature of the group SO(3) which is apparent in this parametrization is that all rotations by the same angle ψ, regardless of the axes about which they occur, belong to the same class in SO(3). This is because $R_{\hat{n}'}(\psi) = R\ R_{\hat{n}}(\psi) R^{-1}$ for an arbitrary rotation R, where $\hat{\mathbf{n}}'$ is the unit vector to which $\hat{\mathbf{n}}$ maps under R [6, p. 95-97]. Whenever an $n \times n$ matrix A can be obtained from another $n \times n$ matrix B as follows: $A = S\ B\ S^{-1}$, where S is any invertible $n \times n$ matrix, A is said to be *similar* to B. Similarity, like isomorphism and conjugation, is an equivalence relation. Thus we can simply say that A and B are similar matrices [10, p. 214,215]. Transformations which are represented by similar matrices actually represent the same transformation in two different coordinate systems, and therefore such transformations always belong to the same class within the group of transformations to which they belong.

Another common parametrization of SO(3) uses the three Euler angles, (α, β, γ). A given rotation R in SO(3) rotates the "fixed frame" Cartesian axes x, y, and z to the "rotated frame" Cartesian axes x′, y′, and z′ [6, p. 97]. The Euler angles determine the orientation of the rotated axes with respect to the fixed axes by three successive rotations. First, the fixed axes are rotated by the angle α about the z axis. Then, the axes obtained from this rotation are rotated by the angle β about the *nodal line*. The nodal line is the line of intersection of the xy and x′y′ planes and is denoted by the unit vector $\hat{\mathbf{N}}$, which lies along it. Finally, the axes obtained from this rotation are rotated by the angle γ about their z axis (which is also the z′ axis). Thus the Euler parametrization can be written in terms of the angle-and-axis parametrization as:

$$R(\alpha, \beta, \gamma) = R_{z'}(\gamma) R_{\hat{N}}(\beta) R_z(\alpha). \tag{6}$$

A given Euler rotation $R(\alpha, \beta, \gamma)$ can be decomposed into rotations about the fixed axes as follows:

$$R(\alpha, \beta, \gamma) = R_z(\alpha) R_y(\beta) R_z(\gamma). \tag{7}$$

This decomposition of R is easier to deal with computationally than the first [6, p. 97,98]. In fact, using this decomposition, the following matrix representation of a given rotation $R(\alpha, \beta, \gamma)$ in SO(3) can be derived

[11, p. 3]:

$$R(\alpha,\beta,\gamma) = \begin{pmatrix} \cos\gamma\cos\alpha - \cos\beta\cos\alpha\sin\gamma & -\cos\gamma\sin\alpha - \cos\beta\cos\alpha\sin\gamma & \sin\gamma\sin\beta \\ \sin\gamma\cos\alpha + \cos\beta\sin\alpha\cos\gamma & -\sin\gamma\sin\alpha + \cos\beta\cos\alpha\cos\gamma & -\cos\gamma\sin\beta \\ \sin\beta\sin\alpha & \sin\beta\cos\alpha & \cos\beta \end{pmatrix}. \quad (8)$$

1.6 The Euclidean Groups

The group E_n, the *Euclidean group*, is the group of linear transformations of the n-dimensional Euclidean space $\mathbf{R}^n$ which preserve the lengths of vectors. In n-dimensional Euclidean space a given vector $\mathbf{x}$ is specified by the coordinates of its endpoint when its base is positioned at the origin. These coordinates are x^i, where $i = 1, 2, ..., n$. The length of a vector $\mathbf{x}$ is given by

$$l = ((x^1)^2 + (x^2)^2 + ... + (x^n)^2)^{\frac{1}{2}}. \quad (9)$$

A transformation in the Euclidean group can include both a translation T($\mathbf{b}$) and a rotation R. The general form of a transformation on $\mathbf{x}$ is:

$$\mathbf{x} \to \mathbf{x}' = R\mathbf{x} + \mathbf{b}, \quad (10)$$

where R is the $n \times n$ matrix which represents the rotation R and $\mathbf{b}$ is the n-dimensional vector $\mathbf{b}$ which defines the translation T($\mathbf{b}$) [6, p. 152].

The Euclidean group E_3 is the symmetry group of motion in physical space ($\mathbf{R}^3$), i.e., the application of any given transformation in E_3 does not alter the laws of motion of a physical system in ($\mathbf{R}^3$). This is because the *Hamiltonian function*, which governs the motion of a physical system, is unaltered by transformations in E_3. The Hamiltonian function H is the sum of the kinetic energy T of the system and the potential energy V of the system. T is the sum of the kinetic energies of all of the particles in the system. In classical physics, the kinetic energy of a single particle is a function of its mass and its speed. Mass is invariant under all spatial transformations. Speed depends on the distance between two points, which is simply the length of a vector from one point to the other, so since vector length is invariant under transformations in E_3, speed is invariant under these transformations. Thus the kinetic energy of a given particle is invariant under transformations in E_3, and therefore T is invariant under these transformations. V is the sum of the potential energies of all of the particles in the system. The potential energy of a single particle is a function of its mass or its charge and of its position. As stated before, mass is invariant under all spatial transformations, as is charge. Position *is* altered by transformations in E_3, but potential energy of a particle does not depend on its absolute position, rather on its position relative to some arbitrarily chosen origin. Since this origin is chosen arbitrarily, we can move it according to the transformation, so that the relative positions of the particles are unchanged. Thus the potential energy of a given particle is invariant under transformations in E_3, and therefore V is invariant under these transformations [6, p. 152,153].

1.7 Groups in 4-dimensional Spacetime: The Lorentz and Poincaré Groups

Spacetime is a four-dimensional real vector space [12, p. 9] that combines space and time into one entity. Like space, spacetime is both homogeneous and isotropic [6, p. 174]. A point in spacetime is called an *event* and has three spatial coordinates, x, y, and z, which specify the event's location in three-dimensional space, and one time coordinate, t, specifying the time at which the event occurs [13]. An event is denoted by $\{x^\mu; \mu = 0, 1, 2, 3\}$, where $x^0 = \mathrm{c}t$, $x^1 = x$, $x^2 = y$, $x^3 = z$, and c is the speed of light in a vacuum [6, p. 174]. Spacetime is equipped with a *metric*, which is a way of specifying the definition of vector length for a vector space [13]. The metric which defines the length of a vector in flat spacetime is known as the *Minkowski metric* and can be written in tensor form as

$$g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Using this metric, the length $|\mathbf{x}|$ of a vector $\mathbf{x}$ in spacetime is defined as follows:

$$|\mathbf{x}|^2 = \sum_{1 \leq \mu < \nu \leq 4} g_{\mu\nu} x^\mu x^\nu, \tag{12}$$

where $g_{\mu\nu}$ denotes the $\mu\nu^{th}$ entry of the tensor $g^{\mu\nu}$ [6, p. 174]. The length of a vector pointing from one event to another in spacetime is known as the *spacetime interval* between the two events.

Homogeneous Lorentz transformations are linear transformations of spacetime which preserve the lengths of vectors in spacetime and map the origin to itself. Because Lorentz transformations preserve vector length, they are represented by matrices Λ with determinant 1 or -1. Those Λ with determinant 1 are analogous to pure rotations in 3-dimensional space, and those with determinant -1 are analogous to rotation-reflections in 3-dimensional space. The operation of a Lorentz transformation Λ on a vector $|\mathbf{x}\rangle$ in spacetime is [6, p. 174,175]:

$$\Lambda|\mathbf{x}\rangle = |\mathbf{x}'\rangle, \tag{13}$$

where

$$x'^{\mu} = \sum_{\nu} \Lambda^{\mu}_{\nu} x^{\nu}. \tag{14}$$

The above equation for x'^{μ} is customarily written simply as

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}, \tag{15}$$

where the summation over ν is implicit. This unwritten, implicit summation is called the Einstein summation convention, which specifies that there is an implicit summation over an index which appears twice. Thus in general, $\sum_i a_i b_i$ can be abbreviated as $a_i b_i$ [14].

A rotation of 3-dimensional space is one type of homogeneous Lorentz transformation. Such a Lorentz transformation Λ_R, which applies the 3-dimensional rotation R to the three spatial coordinates of spacetime and leaves the time coordinate unchanged, is written as:

$$\Lambda_R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & R & \\ 0 & & & \end{pmatrix}, \tag{16}$$

where R is the 3×3 matrix representing the ordinary 3-dimensional rotation R.

Another type of homogeneous Lorentz transformation is called a *Lorentz boost*. Physically, a Lorentz boost is a transformation from a "rest" frame to another frame which is moving at a constant speed along some axis relative to the rest frame. The Lorentz transformation

$$\Lambda_b = \begin{pmatrix} \cosh\xi & \sinh\xi & 0 & 0 \\ \sinh\xi & \cosh\xi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{17}$$

is a Lorentz boost which corresponds to the transformation from a rest frame to another frame moving along the x-axis at speed $v = c \tanh\xi$ relative to the rest frame. Any homogeneous Lorentz transformation can be decomposed into a Lorentz transformation of the form Λ_R and a Lorentz boost. The group of Lorentz transformations of this form is called the *(proper) Lorentz group*, and is denoted $\tilde{\mathrm{L}}_+$ [6, p. 175-177].

The proper Lorentz group consists of all homogeneous transformations of 4-dimensional spacetime (i.e., those mapping the origin to itself) which preserve vector length. The inhomogeneous transformations of 4-dimensional spacetime (i.e., those not mapping the origin to itself) which preserve vector length are known as 4-dimensional translations. As discussed in Section 4, a translation in $\mathbf{R}^4$ by a vector $\mathbf{b}$ is denoted by $\mathrm{T}(\mathbf{b})$. The operation of $\mathrm{T}(\mathbf{b})$ on a vector $|\mathbf{x}\rangle$ in spacetime is:

$$T(\mathbf{b})|\mathbf{x}\rangle = |\mathbf{x}'\rangle, \tag{18}$$

where

$$x'^{\mu} = x^{\mu} + b^{\mu}. \tag{19}$$

The group consisting of the homogeneous transformations of spacetime, spacetime translations, and combinations of these is called the *Poincaré group*, denoted $\tilde{\mathrm{P}}$. As follows from the definition, any transformation in the Poincaré group can be decomposed into a homogeneous Lorentz transformation and a 4-dimensional translation. A general element $g(b, \Lambda)$ of $\tilde{\mathrm{P}}$ is the combination of the Lorentz transformation Λ and the

4-dimensional translation $\mathrm{T}(\mathbf{b})$, and operates on a vector $|\mathbf{x}\rangle$ in spacetime as follows [6, p. 181]:

$$g(b,\Lambda)|\mathbf{x}\rangle = |\mathbf{x}'\rangle, \tag{20}$$

where

$$x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu} + b^{\mu}. \tag{21}$$

1.8 Space Inversion and Time Reversal

In addition to the translations and rotations discussed in the preceding sections, there exists a third type of operation which can be performed on vectors in spacetime while preserving their length. This type of operation is known as *inversion* or reversal, and can invert space or time. There are three types of inversion of 3-dimensional Euclidean space. The first is inversion about one coordinate axis (which inverts the other two axes), the second is inversion about the plane containing two coordinate axes (which inverts the other axis), and the third is inversion about the origin (which inverts all three coordinate axes). The first two inversions can be decomposed into rotations or rotations combined with inversion about the origin. Inversion about the origin is denoted I_s and is represented by the matrix

$$I_s = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{22}$$

The time reversal transformation, denoted I_t, is represented by the matrix

$$I_t = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

It is interesting to note that although the time reversal matrix is simply the space inversion matrix with reversed signs, the physical implications of time reversal are inherently different than those of space inversion. The way in which time reversal transformations are represented also differs considerably from the representations of space inversion transformations [6, p. 221,228,245].

The fact that time reversal is a symmetry transformation for spacetime implies the following: If the time evolution of a given experiment is recorded over a certain period of time, and then a second experiment is carried out on the same system with initial conditions identical to the final conditions of the original experiment expect with linear and angular momenta reversed, then the time evolution of this second experiment will be equivalent to what we would see if we watched the recording of the first experiment in reverse [6, p. 246,247]. Almost all known physical systems obey time reversal symmetry, but one example of a system which does not respect it is the decay of a neutral "K-meson." Systems like this are not yet well understood [6, p. 10]. The implications of time reversal symmetry can be very interesting as they apply to physical properties of systems such as entropy, which does not seem to obey this symmetry. Time reversal symmetry also invokes the question of whether time flows in only one direction. If the flow of time can be reversed without violating the laws of physics, then why should time be restricted to flow in one direction? Questions like these remain open to exploration, and perhaps their solutions will provide further enlightenment as to the nature of the world in which we live, if they are ever attained.

2 Noether's Theorem

In 1915 Emmy Noether, a German mathematician, formulated what is now known as *Noether's theorem*. Her theorem stated that every symmetry transformation of a system (a transformation that preserves the Langrangian of the system) leads to the conservation of some quantity. The essence of the theorem can be seen in the following simplified example. Consider a particle moving in one dimension. The Lagrangian for the particle is defined as $L(q, q')$, where q is the position of the particle and q' is its velocity. The particle's momentum is then defined in terms of the Lagrangian as

$$p = \frac{\partial L}{\partial q'}, \tag{24}$$

and the force acting on the particle is defined as

$$F = \frac{\partial L}{\partial q}. \tag{25}$$

The laws of physics tell us that

$$p' = F. \tag{26}$$

We will now consider a transformation specified by the parameter s which is a symmetry transformation of the system. That is,

$$\frac{d}{ds} L(q(s), q'(s)) = 0, \tag{27}$$

where $q(s)$ is the position of the particle after the transformation and $q'(s)$ is the velocity of the particle after the transformation. The existence of this symmetry transformation leads to the conservation of the quantity

$$C = p \frac{dq(s)}{ds}. \tag{28}$$

This is because

$$C' = p' \frac{dq(s)}{ds} + p \frac{dq'(s)}{ds}, \tag{29}$$

which, substituting from Eqs. (24), (25), and (26) above, becomes

$$C' = \frac{\partial L}{\partial q} \frac{dq(s)}{ds} + \frac{\partial L}{\partial q'} \frac{dq'(s)}{ds}. \tag{30}$$

The above equation is equivalent to

$$C' = \frac{d}{ds} L(q(s), q'(s)), \tag{31}$$

so from Eq. (27),

$$C' = 0, \tag{32}$$

and thus the quantity C is conserved, i.e., it does not change with time. [20]

We now know that the converse of Noether's theorem is also true: every conserved quantity has a corresponding symmetry transformation. This correspondence between symmetries and conserved quantities means that for a given system, the group of symmetry transformations is isomorphic to the group of conserved quantities.

3 Derivation of Conservation Laws from Spacetime Symmetries in Classical Mechanics

It is the symmetries of spacetime which give us conservation laws. Because space is homogeneous, momentum is conserved. Because time is homogeneous, energy is conserved. Together, the conservation of momentum and the conservation of energy allow us to understand and predict much of what happens in the physical world. The connection between the symmetries of spacetime and the conservation laws of physics can be directly seen via the *principle of least action.* The *action* S of a particle is defined for the segment of the particle's path beginning at (x_1,t_1) and ending at (x_2,t_2) as

$$S = \int_{t_1}^{t_2} (T - V)\, dt, \tag{33}$$

where T is the kinetic energy of the particle and V is the potential energy of the particle. The action can be equivalently expressed as

$$S = (T_{avg} - V_{avg})(t_2 - t_1), \tag{34}$$

where T_{avg} and V_{avg} are the time averages of the kinetic and potential energy over the path. For simplicity, we will consider motion in one dimension only. The results obtained here can be generalized to two and three dimensions. The *principle of least action* states that the particle must follow the path from event 1 to event 2 that minimizes the action. This path is unique; that is, there is no other path the particle could follow that would have equal or lesser action. Thus the principle of least action completely describes the path of a particle in a region of known potential, and can be used as an alternative to Newton's laws. Because the principle of least action succinctly expresses the laws of physics for a particle in spacetime, any transformation that preserves the action of a particle is a symmetry transformation of spacetime. [1]

3.1 Conservation of Momentum

We will first show that the action of a free particle is preserved under a spatial translation, and thus that spatial translations are symmetry transformations of spacetime. We will consider a particle moving from event 1 at (x_1,t_1) to event 2 at (x_2,t_2). We will assume that events 1 and 2 are infinitesimally close together, and thus the particle's speed can be approximated to be constant in the region in which they lie. The potential will be constant between the two events as the particle is a free particle with no forces acting upon it. We can let this constant potential be zero. The action of the particle over the path from event 1 to event 2 is then

$$S = T_{avg}(t_2 - t_1), \tag{35}$$

where

$$T_{avg} = \frac{1}{2}mv^2, \tag{36}$$

m is the mass of the particle, and v is the speed of the particle. The particle's speed is given by

$$v = \frac{x_2 - x_1}{t_2 - t_1}, \tag{37}$$

so

$$T_{avg} = \frac{1}{2}m\left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2, \tag{38}$$

and

$$S = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1}. \tag{39}$$

If we now shift both events spatially by a distance Δx, event 1 becomes ($x_1 + \Delta x$,t_1), event 2 becomes ($x_2 + \Delta x$,t_2), and the action becomes

$$S_{shift} = \frac{1}{2}m\frac{[x_2 + \Delta x - (x_1 + \Delta x)]^2}{t_2 - t_1} = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1} = S. \tag{40}$$

Thus the spatial translation did not change the action of the particle along the path. Because the action is additive for segments of the particle's path, the conservation of the action for each infinitesimal segment of the entire continuous path implies that under the spatial translation the action is conserved for the entire path. Thus we conclude that translations in space are symmetry transformations of spacetime. [1]

We will now show that the principle of least action leads to conservation of momentum for a particle moving in one dimension. We will consider a particle moving from event 1 at (x_1,t_1) to event 3 at (x_3,t_3). For a given time t_2 between t_1 and t_3, we wish to determine the particle's position x_2. (As before, we are considering a free particle and events that are infinitesimally close together, and thus we can again allow the potential to be zero in the region where the events lie and the particle's speed to be constant between adjacent events.) The action of the particle over the path from event 1 to event 3 is given by

$$S_{13} = S_{12} + S_{23} = \frac{1}{2}m_{12}\frac{(x_2 - x_1)^2}{t_2 - t_1} + \frac{1}{2}m_{23}\frac{(x_3 - x_2)^2}{t_3 - t_2}, \tag{41}$$

where in general S_{ij} is the action for the segment of the path from event i to event j and m_{ij} is the mass of the particle for the segment of the path from event i to event j. Because space is homogeneous, we can allow x_2 to take on any value. The principle of least action says that x_2 must take the value that gives the least possible action for the path from event 1 to event 3. Thus we must have

$$\frac{dS}{dx_2} = 0 \tag{42}$$

and

$$\frac{d^2S}{dx_2^2} > 0. \tag{43}$$

Eq. (43) is automatically true, because

$$\frac{d^2S}{dx_2^2} = m_{12}\frac{1}{t_2 - t_1} + m_{23}\frac{1}{t_3 - t_2} \tag{44}$$

is always positive, since the particle's mass is positive, t_2 is always greater than t_1, and t_3 is always greater than t_2. Eq. (42) gives

$$m_{12}\frac{x_2 - x_1}{t_2 - t_1} - m_{23}\frac{x_3 - x_2}{t_3 - t_2} = 0 \tag{45}$$

or

$$m_{12}\frac{x_2 - x_1}{t_2 - t_1} = m_{23}\frac{x_3 - x_2}{t_3 - t_2}. \tag{46}$$

This becomes

$$m_{12}v_{12} = m_{23}v_{23}, \tag{47}$$

which is equivalent to

$$p_{12} = p_{23}, \tag{48}$$

where p_{ij} is the momentum of the particle as it travels from event i to event j. Eq. (48) says that momentum must be conserved as the particle moves from the segment of the path between events 1 and 2 to the segment of the path between events 2 and 3. Since this argument will hold for any two adjacent segments of a path, momentum must be conserved throughout any continuous path in spacetime. [1]

We will now show that the homogeneity of space along with the principle of least action leads to conservation of momentum for a particle starting from a point in space x'_1. Due to the principle of least action, we can show that the homogeneity of space leads to the conservation of momentum for any particle starting from any point in space. We will again consider the particle moving from event 1 at (x_1,t_1) to event 2 at (x_2,t_2) to event 3 at (x_3,t_3), with the same assumptions as before. Additionally, we will assume that x'_1 is infinitesimally close to x_1. In order to move from x_1 to x'_1, we must make a spatial translation by the infinitesimal distance dx. The particle starting from position x'_1 will also start at time t_1 and will follow a path identical to (but spatially translated from) the path of the particle starting from position x_1. Thus the new particle will travel from event $1'$ at (x'_1,t_1) to event $2'$ at (x'_2,t_2) to event $3'$ at (x'_3,t_3), where $x'_1 = x_1 + dx$, $x'_2 = x_2 + dx$, and $x'_3 = x_3 + dx$.

We know that the action should be the same for the two paths, because the second path is just a spatial translation of the other. That is,

$$S_{123} = S_{1'2'3'}, \tag{49}$$

where S_{ijk} is the action over the path from event i to event j to event k. Thus we can write

$$\Delta S_{123 \to 1'2'3'} = 0. \tag{50}$$

where

$$\Delta S_{ijk \to lmn} = S_{lmn} - S_{ijk}. \tag{51}$$

Instead of shifting immediately from the path through events 1, 2, and 3 to the path through events $1'$, $2'$, and $3'$, we can shift the path one event at a time. First, we'll shift event 1; that is, we'll go from the path through events 1, 2, and 3 to the path through events $1'$, 2, and 3. Next, we'll shift event 2 (going from the path through events $1'$, 2, and 3 to the path through events $1'$, $2'$, and 3), and finally, we'll shift event 3 (going from the path through events $1'$, $2'$, and 3 to the path though events $1'$, $2'$, and $3'$). The end result of all of this shifting will be the same as shifting directly from the path through events 1, 2, and 3 to the path through events $1'$, $2'$, and $3'$ all at once, so we have the equation

$$\Delta S_{123 \to 1'23} + \Delta S_{1'23 \to 1'2'3} + \Delta S_{1'2'3 \to 1'2'3'} = \Delta S_{123 \to 1'2'3'} = 0. \tag{52}$$

Because the principle of least action tells us that $\frac{dS}{dx_2} = 0$ at the actual value of x_2, and because the shift from x_2 to x_2' is infinitesimal, the shift from the path through events $1'$, 2, and 3 to the path through events $1'$, $2'$, and 3 should not change the action for the path. That is, $\Delta S_{1'23 \to 1'2'3} = 0$, and thus

$$\Delta S_{1'2'3 \to 1'2'3'} = -\Delta S_{123 \to 1'23}. \tag{53}$$

Moreover, because the shifts from x_1 to x_1' and from x_3 to x_3' are also infinitesimal, we can write

$$\Delta S_{123 \to 1'23} = \frac{dS}{dx_1} dx_1, \tag{54}$$

and

$$\Delta S_{1'2'3 \to 1'2'3'} = \frac{dS}{dx_3} dx_3. \tag{55}$$

Thus

$$\frac{dS}{dx_3} = -\frac{dS}{dx_1}, \tag{56}$$

which gives

$$p_{12} = p_{23}. \tag{57}$$

Thus we see that by the homogeneity of space, momentum is conserved from one segment of a path to the next, and thus it is conserved over the entire path. [1]

3.2 Conservation of Energy

We will now show that the action of a particle is preserved under a translation in time, and thus that translations in time are also symmetry transformations of spacetime. We will again consider a particle moving from event 1 at (x_1,t_1) to event 2 at (x_2,t_2), and assume that events 1 and 2 are infinitesimally close together, and thus the potential can be assumed to be constant in the region in which they lie, as can the particle's speed between events 1 and 2. We can let this constant potential be zero. The action of the particle over the path from event 1 to event 2 is

$$S = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1}, \tag{58}$$

as before. If we now shift both events by a change in time Δt, event 1 becomes (x_1,$t_1 + \Delta t$), event 2 becomes (x_2,$t_2 + \Delta t$), and the action becomes

$$S_{shift} = \frac{1}{2}m\frac{(x_2 - x_1)^2}{[t_2 + \Delta t - (t_1 + \Delta t)]} = \frac{1}{2}m\frac{(x_2 - x_1)^2}{t_2 - t_1} = S. \tag{59}$$

Thus the time translation did not change the action of the particle along the path. Because the action is additive for segments of the particle's path, the conservation of the action for each infinitesimal segment of the entire continuous path implies that under the time translation the action is conserved for the entire path. Thus we conclude that translations in time are symmetry transformations of spacetime. [1]

We will now show that the principle of least action leads to conservation of energy for a particle starting from a particular point in space x_1. We will consider a particle moving from event 1 at (x_1,t_1) to event 3 at (x_3,t_3). For a given position x_2 between x_1 and x_3, we wish to determine the time when the particle is at this position. We will call this time t_2. We will again assume that events 1 and 2 and events 2 and 3 are infinitesimally close together, and thus the particle's speed between events 1 and 2 and between events 2 and 3 can be assumed to be constant, as can the potential. We will give the name V_{12} to the constant potential energy of the particle between x_1 and x_2 and the name V_{23} to the constant potential energy of the particle between x_2 and x_3. The action of the particle over the path from event 1 to event 3 is given by

$$S_{13} = S_{12} + S_{23} = \frac{1}{2}m_{12}\frac{(x_2 - x_1)^2}{t_2 - t_1} - V_{12}(t_2 - t_1) + \frac{1}{2}m_{23}\frac{(x_3 - x_2)^2}{t_3 - t_2} - V_{23}(t_3 - t_2). \tag{60}$$

The principle of least action says that t_2 must take the value that gives the least possible action for the path from event 1 to event 3. Thus we must have

$$\frac{dS}{dt_2} = 0 \tag{61}$$

and

$$\frac{d^2S}{dt_2^2} > 0. \tag{62}$$

Eq. (62) is automatically true, because

$$\frac{d^2S}{dt_2^2} = m_{12}\frac{(x_2 - x_1)^2}{(t_2 - t_1)^3} + m_{23}\frac{(x_3 - x_2)^2}{(t_3 - t_2)^3} \tag{63}$$

is always positive, since the mass of the particle is always positive and $t_3 > t_2 > t_1$. Eq. (61) gives

$$-\frac{1}{2}m_{12}\left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2 - V_{12} + \frac{1}{2}m_{23}\left(\frac{x_3 - x_2}{t_3 - t_2}\right)^2 + V_{23} = 0 \tag{64}$$

or

$$\frac{1}{2}m_{12}\left(\frac{x_2 - x_1}{t_2 - t_1}\right)^2 + V_{12} = \frac{1}{2}m_{23}\left(\frac{x_3 - x_2}{t_3 - t_2}\right)^2 + V_{23}. \tag{65}$$

This becomes

$$\frac{1}{2}m_{12}v_{12}^2 + V_{12} = \frac{1}{2}m_{23}v_{23}^2 + V_{23}, \tag{66}$$

which is equivalent to

$$E_{12} = E_{23}, \tag{67}$$

where E_{ij} is the energy of the particle as it travels from event i to event j. Eq. (67) says that energy must be conserved as the particle moves from the segment of the path between events 1 and 2 to the segment of the path between events 2 and 3. Since this argument will hold for any two adjacent segments of a path, energy must be conserved throughout any continuous path in spacetime. [1]

4 Derivation of Conservation Laws from Spacetime Symmetries in Special Relativity

In the theory of special relativity, the action of a free particle is defined for the segment of the particle's path beginning at (x_1, t_1) and ending at (x_2, t_2) as

$$S = -mc^2 \left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}} \Delta t, \tag{68}$$

where m is the rest mass of the particle, c is the speed of light, $v = \frac{x_2 - x_1}{t_2 - t_1}$, and $\Delta t = t_2 - t_1$. Substituting for v and Δt and setting $c = 1$, this becomes

$$S = -m[(t_2 - t_1)^2 - (x_2 - x_1)^2]^{\frac{1}{2}}. \tag{69}$$

In special relativity, a *Lorentz boost* is defined as a shift from one inertial reference frame to another. (An *inertial reference frame* is a non-accelerating, non-rotating frame of reference.) According to the theory of special relativity, the laws of physics are the same as viewed from any inertial reference frame; thus the action of a particle should be invariant under a Lorentz boost. We will let (x_1, t_1) and (x_2, t_2) be the coordinates of events 1 and 2, respectively, in the "laboratory" frame, and (x_1', t_1') and (x_2', t_2') be the coordinates of events 1 and 2 in the frame of a rocket moving with speed v_{rel} relative to the laboratory. In the rocket frame, the action is

$$S' = -m[(t'_2 - t'_1)^2 - (x'_2 - x'_1)^2]^{\frac{1}{2}}. \tag{70}$$

In terms of the laboratory coordinates, the rocket coordinates can be expressed as

$$x'_i = \frac{x_i}{\gamma} - v_{rel} t'_i \tag{71}$$

and

$$t'_i = \frac{t_i}{\gamma} - v_{rel} x'_i, \tag{72}$$

where $\gamma = \frac{1}{(1-v_{rel}^2)^{\frac{1}{2}}}$. Substituting Eqs. (71) and (72) into Eq. (70), we have

$$S' = -m[(t_2 - t_1)^2 - (x_2 - x_1)^2]^{\frac{1}{2}} = S, \tag{73}$$

and thus the action is indeed invariant under a Lorentz boost. Note that the classical action is not invariant under a Lorentz boost (called a *Galilei boost* in classical mechanics). This is because the classical action (and Newton's laws, which can be derived from minimization of the classical action over an arbitrary path) is only an approximation to the true relativistic action. [1]

Lorentz boosts are sometimes referred to as *hyperbolic rotations*, because a Lorentz boost is a rotation in the xt, yt, or zt plane. This type of rotation is different from a spatial rotation in the xy, yz, or xz plane because the time dimension is different from a fourth spatial dimension, as is manifested in the fact that the time coefficient has a negative sign in the Minkowski metric. In special relativity, two observers do not always agree as to whether a given rotation is a purely spatial one or a Lorentz boost, and thus the two types of rotation cannot be separated. In effect, this means that the quantities of energy and momentum cannot be separated in special relativity. Thus the fact that any rotation (purely spatial or Lorentz boost) is a symmetry transformation leads to the conservation of the magnitude of the energy-momentum four-vector in special relativity.

In special relativity, momentum p and energy E are defined as follows:

$$p = m\frac{\Delta x}{\Delta \tau} \tag{74}$$

and

$$E = m\frac{\Delta t}{\Delta \tau}, \tag{75}$$

where $\Delta\tau = (t_2 - t_1)^2 - (x_2 - x_1)^2$. The quantity $\Delta\tau$ is called the *proper time* for the particle's path. Translation in space and translation in time are symmetry transformations in the spacetime of special relativity just as they are in classical spacetime. (This can be verified by checking that the relativistic action is preserved under such transformations.) Conservation of the relativistic momentum can be derived from the homogeneity of space and conservation of the relativistic energy can be derived from the homogeneity of time, both via the principle of least action for special relativity. The derivation proceeds in the same manner as in Section 3 for classical mechanics. [1]

5 The Principle of Least Action in General Relativity

Note that the relativistic action defined above for special relativity does not include a potential energy term. This is because it is the action for a *free* particle, i.e., a particle that has no potential energy because no forces are acting upon it. This coincides with the fact that special relativity only applies to situations in *flat spacetime*. In general relativity, the presence of a massive object causes a curvature of spacetime, which causes other massive objects in the area of curvature to accelerate toward it. In classical mechanics the presence of a gravitational force is represented by a difference in gravitational potential in different locations. In general relativity, gravity is embodied in the curvature of spacetime rather than as a force with a corresponding potential, and thus this curvature takes the place of a gravitational potential. Because of this, it does not make sense to talk about the classical action ($S = \int_{t_1}^{t_2} (T - V)\, dt$) in the context of general relativity.

In general relativity, the *principle of maximal aging* governs the motion of a particle. This principle states that a particle must follow the path from event 1 to event 2 that maximizes the proper time or spacetime interval, $\Delta\tau$, between the two events. (The terms proper time and spacetime interval are interchangeable for timelike intervals, which are the only types of intervals along which a real particle can travel.) The proper time between two infinitesimally separated events in flat spacetime is given by the following equation:

$$(\mathrm{c}d\tau)^2 = (\mathrm{c}dt)^2 - dx^2 - dy^2 - dz^2. \tag{76}$$

The principle of maximal aging is in a sense a more general form of the principle of least action, just as relativistic mechanics are a more general, universally applicable form of Newton's laws. In order to see this, we will show that the principle of maximal aging reduces to the principle of least action in the limit where the speed of the particle under consideration is much less than the speed of light.

5.1 Einstein Field Equations and Schwarzschild Metric

Albert Einstein found, using the principles of conservation of energy and conservation of momentum, that the following tensor equations (Eqs. 77 and 78) describe the geometry of spacetime in the presence of mass:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R, \tag{77}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci curvature tensor, $T_{\mu\nu}$ is the energy-momentum tensor, $g_{\mu\nu}$ is the metric tensor, R is the scalar curvature [15] and

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{78}$$

where the constant κ is defined as follows:

$$\kappa = \frac{8\pi \mathrm{G}}{\mathrm{c}^4}, \tag{79}$$

where G is the universal gravitational constant, and c is the speed of light [16, p. 34]. These equations are known as the *Einstein field equations.* Note that each of the two equations actually represents sixteen separate equations, since each tensor has dimensions 4×4 (μ and ν range from zero to three, representing the four dimensions of spacetime).

Einstein had produced the field equations, but he was unable to find a metric $g_{\mu\nu}$ that satisfied the equations. A year after Einstein had published his theory, Karl Schwarzschild formulated the first exact solution to the field equations. His solution applies to the specific case of a single, spherically symmetric, non-rotating massive body. The metric uses spherical coordinates (ct, r, θ, ϕ) rather than Cartesian coordinates $(\mathrm{c}t, x, y, z)$, as they are more appropriate to the situation. Recall that the Minkowski metric, the metric used to define the spacetime interval between two events in flat spacetime, is defined as follows in terms of the Cartesian coordinates:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{80}$$

The same metric can be expressed in spherical coordinates as

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix} \tag{81}$$

[17]. The *Schwarzschild metric* is expressed as follows in spherical coordinates:

$$g^{\mu\nu} = \begin{pmatrix} 1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{pmatrix}, \tag{82}$$

where M is the mass of the spherically symmetric body, G is the universal gravitational constant, and c is the speed of light [18, p. 186]. The massive body is by default at the origin of the coordinate system. This metric gives the proper time between two events as

$$(\mathrm{c}d\tau)^2 = \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)(\mathrm{c}dt)^2 - \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)^{-1} dr^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2. \tag{83}$$

Note that because of the spherical symmetry in this special case, the metric coefficients do not depend on the spherical coordinates θ and ϕ, and because this is a static situation (a situation which does not change with time) the metric does not depend on time, t. Also because of the spherical symmetry, the elements corresponding to θ and ϕ are the same as in the Minkowski metric [17]. Other metrics have since been developed to describe spacetime in the region surrounding charged and/or rotating black holes [15], but the *Schwarzschild metric* most simply describes spacetime in the vicinity of a black hole. It can also be used to approximate the geometry of spacetime surrounding a planet or star. We will use the geometry of the Schwarzschild metric to show that the principle of maximal aging reduces to the principal of least action in the low velocity limit.

5.2 Deriving the Principle of Least Action

We will consider a particle near the surface of the Earth, moving at a speed much less than the speed of light. As stated above, the proper time for this particle's path is given by

$$(\mathrm{c}d\tau)^2 = \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)(\mathrm{c}dt)^2 - \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)^{-1} dr^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2. \tag{84}$$

With the Earth as the massive body at the center of our coordinate system, the term $\frac{2\mathrm{G}M}{\mathrm{c}^2 r}$ becomes $\frac{0.0089\ \mathrm{m}}{r}$ when we substitute the values $\mathrm{G} \approx 6.7 \times 10^{-11}\ \mathrm{m}^3/(\mathrm{kg}\cdot\mathrm{s}^2)$, $\mathrm{c} \approx 3.0 \times 10^8\ \mathrm{m/s}$, and $M \approx 6.0 \times 10^{24}\ \mathrm{kg}$. Since the particle we are considering is at or beyond the surface of Earth, $r \geq \mathrm{R}$, where R is the radius of the Earth. If $r \geq \mathrm{R} \approx 6.4 \times 10^6\ \mathrm{m}$, then $\frac{2\mathrm{G}M}{\mathrm{c}^2 r} \leq 1.4 \times 10^{-9}$. Since this term is so small, we can make the approximation

$$1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r} \approx 1. \tag{85}$$

However, we will only make this approximation in the second term on the right side of Eq. (75) and not in the first term, because as we will see later it becomes non-negligible in the first term. Eq. (75) thus becomes

$$(\mathrm{c}d\tau)^2 = \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)(\mathrm{c}dt)^2 - dr^2 - (rd\theta)^2 - (r\sin\theta d\phi)^2. \tag{86}$$

Dividing by c^2 and taking the square root on both sides of the equation gives

$$d\tau = \left[\left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r}\right)dt^2 - \frac{dr^2}{\mathrm{c}^2} - \frac{(rd\theta)^2}{\mathrm{c}^2} - \frac{(r\sin\theta d\phi)^2}{\mathrm{c}^2}\right]^{\frac{1}{2}}. \tag{87}$$

Pulling a factor of dt from each term on the right side of the equation, we have

$$d\tau = \left[1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r} - \frac{dr^2}{\mathrm{c}^2 dt^2} - \frac{(rd\theta)^2}{\mathrm{c}^2 dt^2} - \frac{(r\sin\theta d\phi)^2}{\mathrm{c}^2 dt^2}\right]^{\frac{1}{2}} dt. \tag{88}$$

We now make the following substitutions:

$$\frac{dr}{dt} = v_r, \quad r\frac{d\theta}{dt} = v_\theta, \quad r\sin\theta\frac{d\phi}{dt} = v_\phi \tag{89}$$

to get

$$d\tau = \left[1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r} - \frac{1}{\mathrm{c}^2}\left(v_r^2 + v_\theta^2 + v_\phi^2\right)\right]^{\frac{1}{2}} dt. \tag{90}$$

Because v_r, v_θ, and v_ϕ are three orthogonal components of $\mathbf{v}$, where $\mathbf{v}$ is the velocity of the particle, the speed of the particle, v, is given by $v^2 = v_r^2 + v_\theta^2 + v_\phi^2$. Eq. (81) thus becomes

$$d\tau = \left(1 - \frac{2\mathrm{G}M}{\mathrm{c}^2 r} - \frac{v^2}{\mathrm{c}^2}\right)^{\frac{1}{2}} dt. \tag{91}$$

We now see that because we are considering the case in which the speed of the particle is much less than the speed of light, the term $\frac{v^2}{\mathrm{c}^2}$ is also very small, and hence we cannot neglect the term $\frac{2\mathrm{G}M}{\mathrm{c}^2 r}$. Using the approximation

$$(1-x)^n \approx 1 - nx \quad \text{for } x << 1 \text{ and } |nx| << 1, \tag{92}$$

where $x = \frac{2\mathrm{G}M}{\mathrm{c}^2 r} + \frac{v^2}{\mathrm{c}^2}$ and $n = \frac{1}{2}$, we obtain

$$d\tau = \left(1 - \frac{\mathrm{G}M}{\mathrm{c}^2 r} - \frac{v^2}{2\mathrm{c}^2}\right) dt. \tag{93}$$

Now, $\Delta\tau$, the proper time for the particle's entire path, is given by $\Delta\tau = \int d\tau$. Thus the proper time is

$$\Delta\tau = \int \left(1 - \frac{v^2}{2\mathrm{c}^2} - \frac{\mathrm{G}M}{\mathrm{c}^2 r}\right) dt \tag{94}$$

$$\Delta\tau = \int \left[1 - \frac{1}{m\mathrm{c}^2}\left(\frac{1}{2}mv^2 + \frac{\mathrm{G}Mm}{r}\right)\right] dt \tag{95}$$

$$\Delta\tau = \int dt - \int \frac{1}{m\mathrm{c}^2}\left(\frac{1}{2}mv^2 + \frac{\mathrm{G}Mm}{r}\right) dt, \tag{96}$$

where m is the mass of the particle. Recall that the action S is given by

$$S = \int (T - V)\, dt = \int \left(\frac{1}{2}mv^2 + \frac{\mathrm{G}Mm}{r}\right) dt, \tag{97}$$

so

$$\Delta\tau = \Delta t - \frac{S}{m\mathrm{c}^2}. \tag{98}$$

Δt is a constant, since we know the two events between which the particle travels and thus the starting and ending times t_1 and t_2 for its path. Since m and c are also constants for this particle,

$$\Delta\tau = \mathrm{a} - \mathrm{b}S, \tag{99}$$

where a and b are constants. We can thus conclude that for this particle's path, maximization of $\Delta\tau$ is equivalent to minimization of S, and thus the principles of maximal aging and of least action put one and the same constraint on the path of the particle. [19]

6 Generators of Transformations

A very direct manifestation of the connection between symmetry transformations and conservation laws is the group theoretical idea of generators. Recall from section 1.2 that a cyclic group is a group G such that $G = \langle a \rangle$ for some $a \in G$, where the element a is the generator of the group G. Groups of transformations in spacetime are not cyclic, but they also have a *generator* from which all of the elements of the group can be generated. For these groups, the generator is not an element of the group itself; rather, it is an operator whose eigenvalue is a physical quantity. Each element belonging to the group of transformations can be written in terms of this operator, along with some parameter that specifies the individual element.

To demonstrate this idea, we will show how the generator of the group of spatial translations is obtained. We consider an infinitesimal translation $\mathbf{a}$ (that is, a translation infinitesimally separated from the identity operator) and an arbitrary function $f(\mathbf{r})$ of position, $\mathbf{r}$. We define a new function, $F(\mathbf{r})$, as follows:

$$F(\mathbf{r}) = \mathbf{f}(\mathbf{r} - \mathbf{a}). \tag{100}$$

Because $\mathbf{a}$ is infinitesimal,

$$f(\mathbf{r}) = \mathbf{f}(\mathbf{r}) - \mathbf{a} \cdot \nabla \mathbf{f}(\mathbf{r}). \tag{101}$$

The change in the function f, $\delta a f$, due to the translation by $\mathbf{a}$ is then

$$\delta f(\mathbf{r}) = \mathbf{F}(\mathbf{r}) - \mathbf{f}(\mathbf{r}) = -\delta\mathbf{a} \cdot \nabla \mathbf{f} = -\frac{\mathbf{i}}{\boldsymbol{\hbar}} \mathbf{a} \cdot \mathbf{p}\mathbf{f}, \tag{102}$$

because

$$\mathbf{p} = \frac{\boldsymbol{\hbar}}{\mathbf{i}} \nabla. \tag{103}$$

We see that the momentum operator is the generator of translations. Similarly, the angular momentum operator is the generator of rotations in space, and the energy operator, the Hamiltonian, is the generator of translations in time [2]. There is a reciprocal relationship between a group and its generator. Notice that the generator of the group of spatial translations was determined from an infinitesimal translation. The infinitesimal translation was expressed in terms of the momentum operator as $1 - \frac{i}{\hbar}\mathbf{a} \cdot \mathbf{p}$.

A *Lie group* is a "continuously connected group in which the parameters of the product of two elements are continuous, differentiable functions of the parameters of the elements" [4, p. 195]. The generator of a Lie group can always be determined from a transformation that is infinitesimally close to (i.e., continuously connected to) the identity transformation as above. Conversely, each element of the group, including the elements that are finitely separated from the identity element, can be defined in terms of the generator. This is because a finite translation can be generated from the successive application of an infinite number of infinitesimal translations, as follows:

$$\lim_{N \to \infty} \left(1 - \frac{\frac{i}{\hbar}\mathbf{a} \cdot \mathbf{p}}{N} \right)^N = e^{-\frac{i}{\hbar}\mathbf{a} \cdot \mathbf{p}} \tag{104}$$

[3, p. 153]. This follows from the fact that for an infinitesimal translation,

$$1 - \frac{i}{\hbar}\mathbf{a} \cdot \mathbf{p} \approx \mathbf{e}^{-\frac{\mathbf{i}}{\mathbf{\hbar}}\mathbf{a} \cdot \mathbf{p}}, \tag{105}$$

since $e^x \approx 1 + x$ for an infinitesimal x. Thus any spatial translation can be expressed in terms of the momentum operator as $e^{-\frac{i}{\hbar}\mathbf{a} \cdot \mathbf{p}}$. This means that the properties of the generator of a group of transformations determine the characteristics of the group. The generator is a single operator that allows us to make conclusions about the entire group. For example, the fact that the components of the momentum operator commute with each other is a necessary and sufficient condition for the fact that the group of spatial translations is abelian. On the other hand, the fact that the components of the angular momentum operator do not commute with each other is a necessary and sufficient condition for the fact that the group of spatial rotations is not abelian. The structure of the group is in this way analogous to the properties of the generator. [6, p. 84]

7 Conclusion

We have discussed the four major symmetry transformations of flat spacetime. Translation in space corresponds to the conservation of momentum; rotation in space corresponds to the conservation of angular momentum; translation in time corresponds to the conservation of energy; and the Lorentz boost corresponds to the conservation of the magnitude of the energy-momentum four-vector. We have seen that the conserved quantity corresponding to a given symmetry transformation is the quantity which is the eigenvalue of the operator that is the generator of that group of transformations. Finally, we have also seen that the characteristics of a group's generator can provide information about the group itself. The principle of least action and its counterpart in general relativity, the principle of maximal aging, is what has allowed us to see the above correspondence between symmetry transformations and conservation laws.

References

[1] J. Hanc, S. Tuleja, and M. Hancova, Am. J. Phys. **72**, 428 (2004). Also available at http://www.eftaylor.com/pub/Symmetry0104.pdf.

[2] E. Merzbacher, Quantum Mechanics (Wiley & Sons, New York, 1961).

[3] J. S. Townsend, A Modern Approach to Quantum Mechanics (McGraw-Hill, New York, 1992).

[4] L. I. Schiff, Quantum Mechanics (McGraw-Hill, New York, 1955).

[5] J. J. O'Connor and E. F. Robertson, "The development of group theory," from http://www-history.mcs.st-and.ac.uk/history/HistTopics/Development_ group_ theory.html (1996).

[6] W. Tung, Group Theory in Physics (World Scientific, Philadelphia, 1985).

[7] J. J. Rotman, An Introduction to the Theory of Groups (Allyn and Bacon, Boston, 1984).

[8] G. Y. Lyubarskii, The Application of Group Theory in Physics, translated by S. Dedijer (Pergamon Press, New York, 1960).

[9] E. W. Weisstein, "Cyclic Permutation," from MathWorld at http://mathworld.wolfram.com/CyclicPermutation.html (1999).

[10] S. J. Leon, Linear Algebra with Applications, 6th ed. (Prentice Hall, Upper Saddle River, NJ, 2002).

[11] M. Carmeli, Group Theory and General Relativity (McGraw-Hill, New York, 1977).

[12] G. L. Naber, The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity (Springer-Verlag, New York, 1992).

[13] E. F. Taylor and J. A. Wheeler, Exploring Black Holes: Introduction to General Relativity (Addison Wesley Longman, New York, 2000).

[14] E. W. Weisstein, "Einstein Summation," from MathWorld at http://mathworld.wolfram.com/EinsteinSummation.html (1999).

[15] E. W. Weisstein, "Einstein Field Equations," from Eric Weisstein's World of Physics at http://scienceworld.wolfram.com/physics/EinsteinFieldEquations.html (1999).

[16] S. Chandrasekhar, The Mathematical Theory of Black Holes (Oxford University Press, New York, 1992).

[17] R. Gardner, "Relativity and Black Holes: Black Holes" from Bob Gardner's Special Relativity at http://www.etsu.edu/physics/plntrm/relat/general.htm.

[18] R. D'Inverno, Introducing Einstein's Relativity (Clarendon Press, Oxford, 1992).

[19] E. F. Taylor, "Deriving the Nonrelativistic Principle of Least Action from the Schwarzschild Metric and the Principle of Maximal Aging".

[20] J. Baez, "Noether's Theorem in a Nutshell" at http://math.ucr.edu/home/baez/noether.html.