SYMPLECTIC MANIFOLDS, GEOMETRIC QUANTIZATION, AND UNITARY REPRESENTATIONS OF LIE GROUPS

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1. Introduction

Generally speaking, geometric quantization is a scheme for associating Hilbert spaces to symplectic manifolds. In case the manifold is equipped with the Hamiltonian action of a Lie group G the Hilbert space one obtains by quantization should be a unitary representation of G. The notion of geometric quantization was shown to be a powerful representation-theoretic tool when Kostant and Kirillov initiated an orbit method of classifying the irreducible unitary representations of simply-connected solvable Lie groups. This was a generalization of the Borel-Weil theorem to certain classes of noncompact groups. All attempts to generalize this method to arbitrary groups, however, have failed. But in spite of this the orbit method and geometric quantization in general still have important roles to play in representation theory.

This paper attempts to give the definitions and main results of geometric quantization. In the first section we briefly develop the theory of Hamiltonian actions on symplectic manifolds following mainly [2]. The last section deals with geometric quantization itself and is drawn mostly from [5] and [6]. The middle section is a digression on Hamiltonian torus actions culminating in a statement of the celebrated Atiyah [1] and Guillemin-Sternberg [4] convexity theorem.

2. Symplectic and Hamiltonian Actions

Let G be a Lie group with Lie algebra $\mathfrak{g}$ and let G act on the symplectic manifold (W, ω). A vector field ξ on W is said to be *symplectic* if $L_\xi \omega = 0$. Since $L_\xi = di_\xi + i_\xi d$ this is equivalent to requiring that the 1-form $i_\xi \omega$ be closed. A vector field ξ on W is *Hamiltonian* if $i_\xi \omega$ is exact.

Now if ξ and η are symplectic vector fields on W then since $i_{[\xi,\eta]} = [L_\xi, i_\eta]$ we have

$$i_{[\xi,\eta]}\omega = L_\xi i_\eta \omega - i_\eta L_\xi \omega = (di_\xi + i_\xi d) i_\eta \omega - i_\eta (di_\xi + i_\xi d)\omega = di_\xi i_\eta \omega \tag{1}$$

so that $[\xi, \eta]$ is a Hamiltonian vector field.

Associated to any left invariant vector field $X \in \mathfrak{g}$ is a vector field $\underline{X}$ on W defined by

$$\underline{X}_p \phi = \frac{d}{dt}\bigg|_{t=0} (e^{tX}\phi)(p) = \frac{d}{dt}\bigg|_{t=0} \phi(e^{-tX}p).$$

The map $X \mapsto \underline{X}$ is a Lie algebra homomorphism which we call the *infinitesimal action* of G on W. The vector fields $\underline{X}$ are called the *fundamental vector fields* associated to the G-action. Notice that $g_* \underline{X} = \underline{\mathrm{Ad}\, gX}$ for all $g \in G$.

The G-action is said to be *symplectic* if $g^*\omega = \omega$ for all $g \in G$. If G is connected then this is equivalent to the condition that all fundamental fields are symplectic.

Since the symplectic form ω is nondegenerate it provides an identification of $\mathrm{Vect}(W)$ with $\Omega^1(W)$ by $\xi \mapsto i_\xi \omega$. So if $f \in C^\infty(W)$ is any smooth function on W there is a unique Hamiltonian vector field ξ_f, called the *symplectic gradient* of f, such that $i_{\xi_f}\omega = df$. If we define a Poisson

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bracket bracket on W by $\{f,g\} = \xi_f g$ then by (1) we see that that $d\{f,g\} = d(\xi_f g) = d(dg(\xi_f)) = d(i_{\xi_f} i_{\xi_g}\omega) = i_{[\xi_f,\xi_g]}\omega$ so that $\xi_{\{f,g\}} = [\xi_f, \xi_g]$. Thus the map $f \mapsto \xi_f$ is a Lie algebra map.

A symplectic G-action is said to be *Hamiltonian* if there exists a Lie algebra map $\tilde{\mu}\colon \mathfrak{g} \to C^\infty(W)$ such that the following diagram commutes

(2)
$$\begin{array}{ccc} C^\infty(W) & \longrightarrow & \mathrm{Vect}(W) \\ & \overset{\tilde{\mu}}{\nwarrow} & \uparrow \\ & & \mathfrak{g} \end{array} .$$

That is, $d\tilde{\mu}_X = i_{\underline{X}}\omega$ and $\tilde{\mu}_{[X,Y]} = \{\tilde{\mu}_X, \tilde{\mu}_Y\}$. The map $\tilde{\mu}$ is called a *comoment map* for the Hamiltonian G-action.

A *moment map* $\mu\colon W \to \mathfrak{g}^*$ for the G-action is given in terms of the comoment map by $\langle \mu(p), X\rangle = \tilde{\mu}_X(p)$.

Example 1. *Consider the coadjoint action of G on $\mathfrak{g}^*$. Let $\mathcal{O}$ be an orbit. The tangent space of $\mathcal{O}$ is generated by the fundamental vector fields. By definition these fields are given by $\underline{X}_\alpha = -ad^*X(\alpha)$. Because of this, the form ω on $\mathcal{O}$ given by $\omega_\alpha(\underline{X}_\alpha, \underline{Y}_\alpha) = \langle \alpha, [Y,X]\rangle$ is well defined and nondegenerate. Because $g_*\underline{X} = \underline{Ad\,gX}$ it is clear that the G action on $\mathcal{O}$ preserves ω. If we consider $\mathfrak{g} \subseteq C^\infty(\mathfrak{g}^*)$ via the identification $\mathfrak{g} = \mathfrak{g}^{**}$ then we have $(dX)_\alpha \underline{Y}_\alpha = \langle \underline{Y}_\alpha, X\rangle = \langle \alpha, [Y,X]\rangle = \omega_\alpha(\underline{X}_\alpha, \underline{Y}_\alpha)$ so that $i_{\underline{X}}\omega = dX$. Thus, $i_{\underline{X}} d\omega = i_{\underline{X}} d\omega + d i_{\underline{X}}\omega = L_{\underline{X}}\omega = 0$ for all $X \in \mathfrak{g}$. Therefore, ω is a symplectic form and the G-action is Hamiltonian with moment map $\mathcal{O} \hookrightarrow \mathfrak{g}^*$ given by inclusion.*

It is clear by the definition that any two moment maps differ by an element $\alpha \in \mathfrak{g}^*$ such that $\alpha|_{[\mathfrak{g},\mathfrak{g}]} = 0$. In particular, the moment map is unique for semisimple groups. Moreover, the following theorem due to Kostant [6] shows that for semisimple groups there is no distinction between symplectic and Hamiltonian actions.

Theorem 2. *Any symplectic action of a semisimple Lie group G is Hamiltonian.*

Proof. By (1) and the fact that $[\mathfrak{g},\mathfrak{g}] = \mathfrak{g}$ we know that the fundamental vector fields are Hamiltonian. We can therefore chose an arbitrary lift $\lambda\colon \mathfrak{g} \to C^\infty(W)$ such that $d\lambda_X = i_{\underline{X}}\omega$. This lift can then be made into a Lie algebra homomorphism by adding a suitable element of $\mathfrak{g}^*$. □

Another fact that follows easily from definitions is that the induced map $\mu_*\colon TW \to \mathfrak{g}^*$ is the transpose of the map $\mathfrak{g} \to T^*W$ given by $X \mapsto i_{\underline{X}}\omega$. Thus, $\ker \mu_*$ is the ω-orthogonal of the tangent space to the G-orbit. Hence, the rank of μ at $p \in W$ is equal to the dimension of $G \cdot p$. Since $\dim G\cdot p = \dim G - \dim G_p$ we see that μ is a submersion at $p \in W$ if and only if G_p is discrete. Moreover, it follows that μ has zero rank at all fixed points of G.

If the group is connected then the moment map is G-equivariant with respect to the coadjoint action of G on $\mathfrak{g}^*$. This follows from the fact that the respective fundamental vector fields are μ-related.

3. Torus Actions

By the previous discussion we know that the fixed points of a Hamiltonian action correspond to certain critical points of the moment map. Furthermore, the equivariant tubular neighborhood theorem tells us that this set of fixed points is a submanifold of W. These two facts suggest that Morse theory might have a role to play in analyzing Hamiltonian torus actions.

Let (W,ω) be a symplectic manifold endowed with a Hamiltonian action of a torus T with moment map $\mu\colon W \to \mathfrak{t}^*$. Let $X \in \mathfrak{t}$ generate T in the sense that e^{tX} is dense in T. Let $H\colon W \to (\mathbb{R}X)^* = \mathbb{R}$ be defined by $H = i^* \circ \mu$ where $i\colon \mathbb{R}X \to \mathfrak{t}$ is inclusion. It follows by definition that $H = \tilde{\mu}_X$ so that $\underline{X}$ is the Hamiltonian field associated to H. That is, $i_{\underline{X}}\omega = dH$ so that the critical points of H are the zeros of $\underline{X}$ or, as X generates T, the fixed points of T. The function H is called an *almost periodic Hamiltonian.*

With the help of a Riemannian metric one derives a calibrated almost complex structure J and a hermitian metric. Of course the T-action is symplectic so if the Riemannian metric is chosen to be T-invariant then T will preserve J and the Hermitian form. Thus if $z \in W$ is a fixed point of T we see that T acts unitarily on the complex vector space T_zW, that is to say, as a subgroup of $U(n)$. Notice that all these transformations are diagonalizable: the exponential map $\mathfrak{t} \to T$ is onto and $\mathfrak{t}$ sits inside $\mathfrak{u}(n)$ (the skew-Hermitian matrices). So as T is commutative there is a basis of T_zW in which all the elements of T are diagonalizable. This allows us to give an eigenspace decomposition

$$T_zW = V_0 \oplus V_1 \oplus \cdots \oplus V_k \tag{3}$$

where V_0 is the subspace of fixed points of T.

Since the Riemannian metric is T-invariant the map $\exp_z\colon T_zW \to W$ is equivariant. Thus if Z is the submanifold of fixed points we see that $T_zZ = V_0$. We have therefore proven

Theorem 3. *The set of fixed points Z of a Hamiltonian torus action is a symplectic submanifold of (W,ω).*

Proof. Indeed, we have shown that Z is an almost complex submanifold of W. Thus, for any $\xi \in T_pZ$ we have $J\xi \in T_pZ$ so that $\omega(\xi, J\xi) \neq 0$ since the almost complex structure is ω-calibrated. $\square$

Since $X_H \in \mathfrak{t} \subseteq \mathfrak{u}(n)$ we see that e^{X_H} acts on V_j by multiplication by some scalar $e^{-i\lambda_j}$ for $\lambda_j \in \mathbb{R}$. Further, since X_H generates T we have that λ_j is nonzero for $j \neq 0$. If $v_0^1, \ldots, v_0^r, v_1, \ldots, v_k$ are local coordinates corresponding to the decomposition (3) and if $v_j = p_j + iq_j$ then as $\xi_H = \underline{X_H}$ we have that

$$\xi_H = \sum_{j=1}^{k} \lambda_j \left(q_j \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial q_j}\right) \tag{4}$$

in these coordinates. With the help of the equivariant Darboux theorem there is a neighborhood U of $0 \in T_zW$ and an equivariant map $\phi : U \to W$ such that $\phi^*\omega$ is the constant form $\sum dp_j \wedge dq_j$. So from (4) it follows that $dH = \sum_{j=1}^k \lambda_j (p_j dp_j + q_j dq_j)$ so that $H = H(z) + \frac{1}{2}\sum_{j=1}^k \lambda_j (p_j^2 + q_j^2)$ in these coordinates.

From this local formula for H it follows that its second derivative H_{**} is diagonal with each λ_j occurring twice so that it is nondegenerate on $V_1 \oplus \cdots \oplus V_k$ (i.e., transverse to Z) and its index (the number of negative diagonal entries), is even. Thus, we have shown

Theorem 4. *An almost periodic Hamiltonian H associated to a Hamiltonian torus action is a Morse-Bott function all the critical submanifolds of which have even index.*

Knowing this we can apply the standard techniques of Morse theory to analyze Hamiltonian torus actions. For instance, we can show fairly easily one of the key results of [1]:

Theorem 5. *If H is an almost periodic Hamiltonian associated to a torus action on a compact symplectic manifold (W,ω), then H has a unique locally minimal submanifold and a unique locally maximal submanifold; moreover, all of its nonempty levels $H^{-1}(t)$ are connected.*

Proof. The key step in the proof is to show that all of the subsets $W_a = \{p \in W \mid H(p) \leq a\}$ are connected. If some W_a is not connected then in order to connect all the pieces (as W is connected) we must pass through a critical level for which the sphere bundle of the negative normal bundle is disconnected. This can only happen if the index of the critical level is 1, which it cannot be. □

Because of the local description of the action given above in (3) we see that the connected components of the fixed submanifold must be separated. Thus, if W is compact there can be only finitely many such components. Since the moment map has rank zero on the fixed set we know that the images of these components under the moment map must be single points. Tying this all together is the convexity theorem of Atiyah [1] and Guillemin-Sternberg [4].

Theorem 6. *Let (W, ω) be a compact symplectic manifold with Hamiltonian torus action. Let $Z_1, \ldots, Z_m$ be the connected components of the fixed submanifold and let $\mu(Z_j) = c_j \in \mathfrak{t}^*$. The image of the moment map is the convex hull of the points $c_1, \ldots, c_m$.*

4. Geometric Quantization

The theory of Hamiltonian actions plays an important role in the theory of unitary representations of Lie groups. In this section, as usual, we let (W, ω) be a symplectic manifold with Hamiltonian action of a connected Lie group G.

If ω is a symplectic form on W then it defines a DeRham cohomology class $[\omega]$ in $H^2(W)$. The form is said to be *integral* if $[\omega]$ is in the image of the Weil homomorphism $\check{H}^2(W, \mathbb{Z}) \to H^2(W)$. Kostant shows in [6] that ω is integral if and only if there exists a complex line bundle L over W with connection ∇ and ∇-invariant Hermitian structure h such that $\omega = \operatorname{curv}(L)$. Moreover, the number of such bundles, up to equivalence, is classified by the character group of $\pi_1(W)$. The data (L, ∇, h) are called *prequantum data* on W.

There is a canonical representation of $\mathfrak{g}$ on smooth sections of L given by

$$X \cdot s = \nabla_{\underline{X}} s + 2\pi i \tilde{\mu}_X s. \tag{5}$$

The prequantum data is said to be *G-invariant* if there is a lifting of the G-action to L such that the induced action on sections is given by (5). A theorem of Palais [7] shows that such a lifting always exists if G is simply connected.

In order to pass from prequantization to quantization one needs the concept of polarization.

Definition 7. *A* polarization *of a symplectic manifold (W, ω) is an integrable complex Lagrangian subbundle F of the complexified tangent bundle $TW \otimes \mathbb{C}$. The polarization is said to be* positive-definite *if the Hermitian form $i\omega(v, \bar{w})$ is positive definite on each fiber F_p.*

Now if a symplectic manifold (W, ω) is equipped with G-invariant prequantum data (L, ∇, h) and a polarization F we say that a smooth section $s\colon W \to L$ is *holomorphic* if $\nabla_\zeta s = 0$ for all vector fields ζ taking values in the conjugate bundle $\bar{F}$.

Our use of the word "holomorphic" is not coincidental as the following theorem shows.

Theorem 8. *Any symplectic manifold possessing a positive-definite polarization is Kähler.*

Proof. Since the form $i\omega(v, \bar{w})$ is positive-definite on F it follows that $F \cap \bar{F} = 0$. Because of this fact we can define an almost complex structure J on W such that $F_p = \{v - iJv \mid v \in T_pW\}$: just set $Jv = w$ whenever $v - iw \in F_p$. Since F is Lagrangian we have $\omega_p(v - iJv, w - iJw) = 0$ for all $v, w \in T_pW$. Evaluating the real and imaginary parts of this equation we see that J preserves the symplectic form and that the bilinear form $\omega_p(Jv, w)$ is symmetric. Since $i\omega(u, \bar{u}) = 2\omega(Jv, v)$ for

$u = v - iJv \in F$ we see that the form $\omega_p(Jv, w)$ is positive-definite. That is, J is calibrated to ω. Because F is integrable we see that the almost complex structure is actually complex. □

This theorem shows that a positive-definite polarization F equips W with a complex structure for which the vectors in F are the holomorphic tangent vectors. Because of this we sometimes say that a polarization F is *Kähler* if $F \cap \bar{F} = 0$.

Let S denote the space of all sections of L and let S_F denote the space of all holomorphic sections. The G-action on S is given by $gs(p) = g(s(g^{-1}p))$.

Theorem 9. *If the polarization F is G-invariant then the G-action preserves S_F.*

Proof. Since the G-action on L is connection preserving we know ([6], Lemma 1.11.2) that for any section s and any $\xi \in T_pW \otimes \mathbb{C}$ we have $\nabla_\xi(gs) = g(\nabla_{g_*^{-1}\xi} s)$. The result follows from this formula and the fact that F (and therefore $\bar{F}$) is G-invariant. □

Hence, we obtain a representation of G on the space of all holomorphic sections. If W is compact then by using the hermitian form h on L and the Liouville measure on W we obtain a Hilbert space structure on S_F by

$$||s||^2 = \int_W |\langle s, s\rangle|^2. \tag{6}$$

As h is G-invariant this representation is seen to be unitary. In case W is noncompact we cannot simply use the Liouville measure to give S_F a Hilbert space structure, but must instead use half-densities.

This method of associating of a unitary representation to a Hamiltonian action which we have just described is what we call geometric quantization. When the Lie group is compact and the symplectic manifolds under consideration are the coadjoint orbits then geometric quantization is the same as the Borel-Weil theory. In particular, all irreducible unitary representations of G in this case arise by quantizing coadjoint orbits. It was the insight of Kirillov and Kostant to apply geometric quantization to the noncompact case. This approach led to the following theorem of Auslander and Kostant [3].

Theorem 10. *Let G be a simply-connected solvable Lie group and $\mathcal{O}$ an integral coadjoint orbit (so that $\mathcal{O}$ possesses a G-invariant prequantization). Then this orbit can be quantized to obtain an irreducible unitary representation of G. Moreover, if G is of Type I then any irreducible unitary representation can be obtained in this way.*

References

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