BRAID GROUP REPRESENTATIONS

A Thesis

Presented in Partial Fulfillment of the Requirements for
the Degree Master of Science in the
Graduate School of the Ohio State University

By
Craig H. Jackson, B.S.

* * * * *

The Ohio State University
2001

Approved by

Master's Examination Committee:

Dr. Thomas Kerler, Advisor

Dr. Henry Glover

Advisor
Department Of Mathematics

ABSTRACT

It is the purpose of this paper to discuss representations of the braid groups and some of the contexts in which they arise. In particular, we concentrate most of our attention on the Burau representation and the Krammer representation, the latter of which was recently shown to be faithful ([3], [14]). We define these representations as actions of B_n on the homology of certain covering spaces. Explicit matrices for these representations are then calculated.

We also develop the theory of braided bialgebras and show how representations of the braid groups arise in this context in a systematic way. It is proved that both the Burau representation and the Krammer representation are summands of representations obtained from the quantum algebra $U_q(\mathfrak{sl}_2)$.

Calling attention to the important connection between braids and links, we use geometrical methods to show that the Burau matrix for a braid can be used to find a presentation matrix for the Alexander module of its closure. Hence, the Alexander polynomial of a closed braid is shown to arise from the Burau matrix of the braid.

In the final chapter we consider representations of B_n arising from the Hecke algebra. This is put into a tangle theoretic context which allows for the extension of these representations to string links. It is also shown how the Conway polynomial arises from the constructions of this chapter.

Preceding all of this, however, is a survey of many of the important results of braid theory. The emphasis throughout is on making the concepts and results clear and approachable. In general, geometrical methods are emphasised.

To Mark Burden

ACKNOWLEDGMENTS

I wish to thank my advisor, Thomas Kerler, for the many ideas he gave me and for his suggestion that I write this thesis.

Thanks also to Tom Stacklin for giving me a copy of the osuthesis document class which I used to typeset this paper.

VITA

1975	Born in Columbia, Missouri
1999	B.S. in Mathematics, The University of Alaska, Anchorage
1999-Present	Graduate Teaching Associate, The Ohio State University

FIELDS OF STUDY

Major Field: Mathematics

Specialization: Topology

TABLE OF CONTENTS

LIST OF FIGURES

CHAPTER 1
INTRODUCTION

Emil Artin [1] first introduced the braid groups B_n in 1925. He also proved many of the most fundamental results concerning them. He gave a finite presentation of B_n and solved the word problem for these groups.

Perhaps the longest standing open question concerning the braid groups is whether or not they are linear. That is, does there exist a faithful representation of B_n into a group of matrices over a commutative ring. Interest in this problem originated with the discovery by Burau [5] in 1935 of a nontrivial n-dimensional linear representation of B_n. This representation was long considered a candidate for faithfulness. In fact, there are simple arguments which show this representation to be faithful for $n \leq 3$ ([4], Thm. 3.15). However, J. Moody [18] discovered in 1991 that the Burau representation is unfaithful for $n \geq 9$. Stephen Bigelow [2] later improved this result to $n \geq 5$. The faithfulness of the Burau representation in the case $n = 4$ is still unknown.

The recent work of Daan Krammer and Stephen Bigelow, however, has given an answer to the question of the linearity of the braid groups. In [13] Krammer considered a two-variable representation of B_n first constructed by Lawrence [15] and showed it to be faithful for $n = 4$. Soon after, Bigelow [3] used topological techniques to prove the faithfulness of this representation for all n. Extending the results of

his earlier paper, Krammer [14] then gave another proof of this result using different methods.

These results have sparked renewed interest in braid groups and their representations. Recent papers by Lin/Tian/Wang [17] and Silverman/Williams [19] give generalizations of the Burau representation to the semigroup St_n of string links. One might similarly hope to find extensions of Krammer's representation to more general objects such as string links or tangles. Additionally, a particularly exciting result would be to discover a use for the Krammer representation in constructing link invariants.

The Burau representation itself is closely related to the Alexander polynomial link invariant as they both may be constructed via certain infinite cyclic coverings which are themselves closely related, one being a subcover of the other. In fact, for an open book decomposition of a knot the Alexander polynomial is explicitly given by the determinant of the (reduced) Burau matrix minus the identity ([4], Thm. 3.11):

$$(1 + t + \cdots + t^{n-1})\Delta_{\hat{\beta}}(t) = \det(\psi_{\mathrm{r}}\beta - I) \tag{1.1}$$

It is known that the Alexander polynomial separates knots with a 3-braid open book decomposition but not for more complicated knots.

One of the original motivations for this thesis was to find a link invariant that is related to the Krammer representation in somewhat the same way as the Alexander polynomial is related to the Burau representation. Due to the faithfulness of the representation one would expect this to be a very powerful invariant.

To this end we carefully investigate in this thesis the relation between the Alexander polynomial and Burau representation using both the relation between the covering spaces used to define them as well as the categorical picture of tangles. This latter approach has proven to be very effective in the definition of many other knot polynomials in recent times.

Particularly, we give a proof of the relation (1.1) without using the usual Fox

differential calculus of fundamental groups. Instead we consider cyclic coverings of tangle complements that we glue together and compute the homology of.

In the last section we consider a fibre functor on a special tangle category. This functor that we define is a representation of the tangle category to the category of complex vector spaces. We will see that the spaces associated to the objects in the tangle category are the Hecke algebras of type A_n. In particular we show that any representation of the braid group which arises by passing through the Hecke algebra will extend to the semigroup of string links. This result contains the extension of the Burau representation to string links given by Lin [17] as a corollary. In addition, we point out how our constructions in this chapter encompass the Conway-normalized Alexander polynomial in their definition.

The Burau and Krammer representations of B_n are closely related to the first two in a series of representations of the braid group constructed by Lawrence. These representations are given via local coefficient systems on the configuration space of m $(= 1, 2, \dots)$ points in an n-punctured plane. These representations factor though the Hecke algebra which is known to be related to the HOMFLYPT knot polynomial. Similar local systems and their homology have later been related by Felder [7] to a quantum group action, in particular the quantum group $U_q(\mathfrak{sl}_2)$, where the E and F generators can be thought of as acting by adding or deleting points in the configuration space – i.e., paths in the local system – and in particular commute with the representation of the braid group.

A result of this paper is to make this relation between the Krammer representation and $U_q(\mathfrak{sl}_2)$ more precise by a computation. Namely, after developing the general theory of braided bialgebras and the braid group representations that they induce we show with explicit computations that both the Burau and Krammer representations are equivalent to summands of one of these induced representations. Since $U_q(\mathfrak{sl}_2)$ is used to define the Jones polynomial and, more generally, tangle functors we anticipate this to be of use in constructing link invariants from the Krammer representation.

Note that Zinno [22] also gives a relation of the Krammer representation with a quantum representation. Only here he uses a representation associated to the Birman-Wenzl-Murakami algebra $C_n(\alpha, l)$. By identifying parameters $q = -\alpha^{-2}$ and $t = \alpha^3 l^{-1}$ he shows the Krammer representation to be equivalent to the simple representation of the BMW algebra associated to the Young diagram with one row of $n-2$ boxes (see [22] or [20]). Generically, $C_n(\alpha, l)$ is the centralizer of $U_q(\mathfrak{g})$ where $\mathfrak{g}$ is a Lie algebra of type C. There are some contraints on the parameter depending on the rank of $\mathfrak{g}$ which essentially means that all ranks – or even continuous ones – have to be considered. In our description, using $\mathfrak{sl}_2$, we only need to consider rank 1 instead of continuous rank at the price of admitting continuous highest weights.

The majority of Chapter 2 is concerned with presenting results which are basic to braid theory. In Chapter 3 we give in detail the homological definitions of the Burau and Krammer representations and calculate their matrices. Chapter 4 is concerned with the relation between the Burau representation and the Alexander polynomial. In Chapter 5 we give the calculations that relate the Burau and Krammer representations to the braid group representations induced by the quantum algebra $U_q(\mathfrak{sl}_2)$. Finally, in Chapter 6 we investigate the tangle category and the representations arising from it.

CHAPTER 2

THE BRAID GROUPS B_n

In this chapter we give a brief overview of the basic definitions and results of braid theory. Our exposition will mainly follow [4] leaving out some of the proofs unless they are important for later results.

2.1 First Definitions

Let D be an oriented disk in the complex plane. Let

$$C = \{(z_1, \ldots, z_n) \in D^n \mid z_i \neq z_j \text{ if } i \neq j\}$$

where D^n is the Cartesian product of n copies of D. The symmetric group S_n on n letters acts on C by $\phi(z_1, \ldots, z_n) = (z_{\phi(1)}, \ldots, z_{\phi(n)})$ Let $\hat{C} = C/S_n$ be the identification space of C under this action. $\hat{C}$ is then the collection of all unordered n-tuples $\{z_1, \ldots, z_n\}$ of elements of D such that $z_i \neq z_j$ if $i \neq j$. The projection $C \to \hat{C}$ is a regular n-fold covering map.

Definition 1. *The fundamental group $\pi_1(\hat{C})$ of the space $\hat{C}$ is called the* braid group *on n strands and is denoted by B_n.*

A geometrically more intuitive picture of B_n can be obtained as follows. Choose a base point $\hat{z}^0$ in $\hat{C}$. Let $z^0 = (z_1^0, \ldots, z_n^0)$ be a lift of $\hat{z}^0$ to C. Any element in $\pi_1(\hat{C}, \hat{z}^0)$ is represented by a loop $\hat{\alpha} : (I, \{0, 1\}) \to (\hat{C}, \hat{z}^0)$ which lifts to a unique path $\alpha : (I, \{0\}) \to (C, z^0)$. Moreover, since $\hat{\alpha}(1) = \hat{z}^0$ we have $\alpha(1) = \phi z^0$ for some ϕ in S_n. We may write $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_i is a path in C connecting

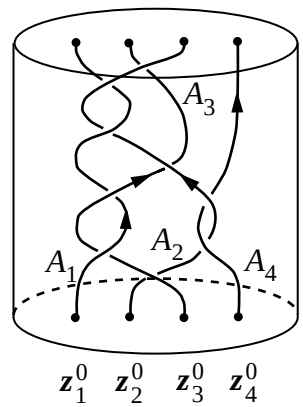


Figure 2.1: A geometric braid on 4 strands.

z_i^0 to $z_{\phi(i)}^0$. These coordinate functions α_i define arcs $A_i = (\alpha_i(t), t)$ in $D \times I$. Since $\alpha(t) \in C$ for all t, the arcs A_i are disjoint. The union $\beta = A_1 \cup \cdots \cup A_n$ is called a *geometric braid* on n strands. See Figure 2.1.

A homotopy of loops $\hat{\alpha}, \hat{\alpha}' : I \to \hat{C}$ relative to $\hat{z}^0$ lifts to a homotopy F of paths $\alpha, \alpha' : I \to C$ relative to $\{z^0, \phi z^0\}$. That is, we have a continuous map $F : I \times I \to C$ such that

$$F(t, 0) = \alpha(t)$$

$$F(t, 1) = \alpha'(t)$$

$$F(0, s) = z^0 = (z_1^0, \ldots, z_n^0)$$

$$F(1, s) = \phi z^0 = (z_{\phi(1)}^0, \ldots, z_{\phi(n)}^0)$$

Let β and β' be the geometric braids defined by $\hat{\alpha}$ and $\hat{\alpha}'$, respectively. Then the homotopy F defines a continuous family of geometric braids β_s such that $\beta_0 = \beta$ and $\beta_1 = \beta'$. Any two geometric braids are said to be *equivalent* if such a continuous family β_s exists and we call the family β_s an *equivalence* of β with β'.

Any geometric braid $\beta = A_1 \cup \cdots \cup A_n$ with $A_i = (\alpha_i(t), t)$ gives an element $\hat{\alpha}$ of $\pi_1(\hat{C})$ in an obvious way. Similarly, an equivalence of β with β' gives a homotopy

of $\hat{\alpha}$ with $\hat{\alpha}'$. Hence, the collection of geometric braids modulo equivalence gives an alternate description of B_n where composition of loops in $\pi_1(\hat{C})$ corresponds to concatenation of geometric braids. Because of this, from here on we write B_n to denote both $\pi_1(\hat{C})$ and the group of geometric braids on n strands.

Since we are presently in the business of giving definitions, we give here the definition of the semigroup St_n of n-string links. This is a definition that will be needed in Chapter 6. String links are just like braids except we allow the strands to move up and down. To be precise, an *n-string link* is defined, up to isotopy, to be *any* collection of n arcs $A_i : I \to D \times I$ such that $A_i(0) = z_i^0$ and $A_i(1) = z_{\phi(i)}^0$ for some $\phi \in S_n$. The point is that for string links the arcs A_i may be knotted whereas they can not be in the case of braids. The multiplication in St_n is given by concatenation and. The identity element of St_n is the same as the identity of B_n.

2.2 A Presentation of B_n

Let $\sigma_1, \ldots, \sigma_{n-1} \in B_n$ be the braids defined in Figure 2.2. It should be clear that $\sigma_1, \ldots, \sigma_{n-1}$ generate B_n and satisfy the following relations:

$$\sigma_i \sigma_j = \sigma_j \sigma_i \qquad |i - j| \geq 2 \tag{2.1}$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad 1 \leq i \leq n-2 \tag{2.2}$$

In fact, the following theorem of E. Artin shows that these generators and relations are sufficient to define B_n.

Theorem 2. *The braid group $B_n = \pi_1(\hat{C})$ admits a presentation with generators $\sigma_1, \ldots, \sigma_{n-1}$ and relations given by* (2.1)–(2.2).

For more details see [4], page 18.

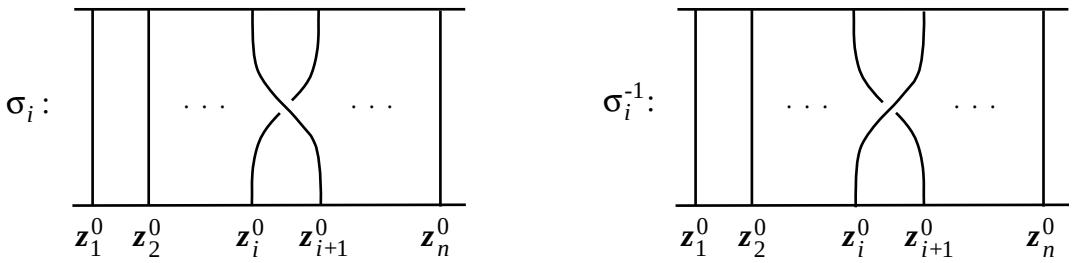


Figure 2.2: Generators of B_n.

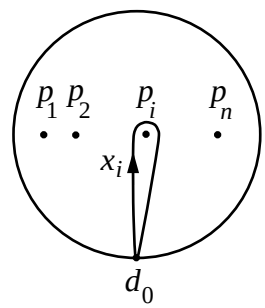


Figure 2.3: The space D_n and the generators of its fundamental group.

2.3 B_n and Aut(F_n)

Fix a set P of n distinct points $p_1, p_2, \ldots, p_n$ in the interior of the unit disk D and let d_0 be a point on the boundary of D. Let D_n denote the n-punctured disk $D - P$. Let $d_0 \in \partial D_n$ be a base point. The fundamental group of D_n is isomorphic to F_n, where $F_n = \langle x_1, x_2, \ldots, x_n \rangle$ is the free group on n letters. The generators $x_1, x_2, \ldots, x_n$ correspond to loops encircling the punctures $p_1, p_2, \ldots, p_n$, respectively (see Figure 2.3).

Let $h : D_n \to D_n$ be a homeomorphism of D_n fixing ∂D_n pointwise. This induces an auotmorphism h_* of the fundamental group $\pi_1(D_n, d_0) = F_n$. Let M_n denote the

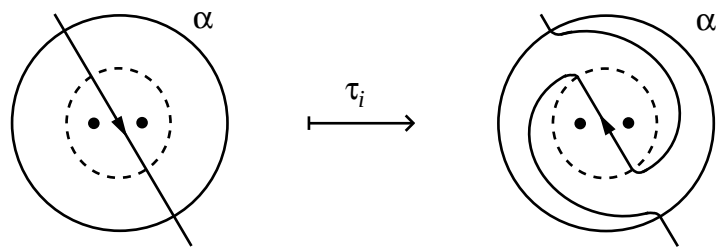


Figure 2.4: The Dehn half-twist.

subgroup of $\mathrm{Aut}(F_n)$ consisting of all such automorphisms h_* where h is a homeomorphism of D_n fixing ∂D_n pointwise. We wish to show $B_n \cong M_n$.

Let $h_* \in M_n$. The homeomorphism h extends uniquely to a homeomorphism $\bar{h}$ of D to itself which permutes P. This map $\bar{h}$ is isotopic to 1_D through an isotopy that fixes the boundary of D_n (see [4], Lemma 4.4.1). Let $F : D \times I \to D$ be such an isotopy. Then for each $t \in I$, $F(x,t)$ is a homeomorphism of D to itself such that $F(x,0) = 1_D$ and $F(x,1) = \bar{h}$. Define $\bar{F} : D \times I \to D \times I$ by $\bar{F}(x,t) = (F(x,t),t)$. The image of $P \times I$ under $\bar{F}$ is a geometric braid. This gives a well defined map $\mathrm{Aut}(F_n) \to B_n$. To give a map in the other direction we need to define the so called "Dehn half-twists."

Consider a simple closed curve α enclosing the punctures p_i and p_{i+1}. Identify the region enclosed by α with the twice-punctured disk $D_2 = D - \{1/4, -1/4\}$. Identify the annulus $A = \{z \in D_2 \mid 1/2 \leq |z| \leq 1\}$ with $S^1 \times I$. The *Dehn half-twist* τ_i is defined as follows: τ_i is the identity outside of D_2; τ_i sends (s,t) to $(e^{-\pi i t}s, t)$ for $(s,t) \in S^1 \times I$; and τ_i is rotation by an angle of $\pi/2$ on $\{z \in D_2 \mid |z| \leq 1/2\}$. See Figure 2.4.

The action on F_n of $(\tau_i)_*$ (which we also denote by τ_i) is given by

$$\tau_i x_i = x_i x_{i+1} x_i^{-1} \tag{2.3}$$

$$\tau_i x_{i+1} = x_i \tag{2.4}$$

$$\tau_i x_j = x_j \qquad j \neq i, i+1. \tag{2.5}$$

which can easily seen by drawing a few diagrams. From these relations we see immediately that $\tau_i\tau_{i+1}\tau_i = \tau_{i+1}\tau_i\tau_{i+1}$ for all $i = 1, 2, \ldots, n-1$, and that $\tau_i\tau_j = \tau_j\tau_i$ for $|i-j| \geq 2$. That is, the elements $\tau_1, \tau_2, \ldots, \tau_{n-1}$ satisfy the braid relations (2.1) and (2.2). Hence, we have a homomorphism $B_n \to \mathrm{Aut}(F_n)$ given by $\phi_i \mapsto \tau_i$.

These homomorphisms $\mathrm{Aut}(F_n) \to B_n$ and $B_n \to \mathrm{Aut}(F_n)$ are inverse to one another. This proves the following theorem.

Theorem 3. *The homomorphism $\phi_i \mapsto \tau_i$ gives a faithful representation of the braid group B_n as a group of automorphisms of the free group $F_n = \langle x_1, x_2, \ldots, x_n \rangle$. This homomorphism induces an isomorphism of B_n with the subgroup $M_n \leq Aut(F_n)$ of all automorphisms of F_n induced by homeomorphisms $h : D_n \to D_n$ which fix ∂D_n pointwise.*

Because of this theorem we may identify B_n with M_n by setting ϕ_i equal to τ_i and so consider B_n to be contained in $\mathrm{Aut}(F_n)$.

We end this section with a theorem due to Artin.

Theorem 4. *Let β be an automorphism of F_n, then $\beta \in B_n \subseteq Aut(F_n)$ if and only if β satisfies the conditions*

$$\beta x_i = w_i x_{\mu_i} w_i^{-1} \qquad 1 \leq i \leq n \tag{2.6}$$

$$\beta(x_1 x_2 \ldots x_n) = x_1 x_2 \ldots x_n \tag{2.7}$$

where $(\mu_1, \ldots, \mu_n)$ is a permutation of $(1, 2, \ldots, n)$ and w_i is any element of F_n.

The necessity of conditions (2.6) and (2.7) follows immediately from (2.3)–(2.5). For sufficiency see [4], page 30.

Let $\beta \in B_n$ be a braid with β acting on F_n as in (2.6) – (2.7). Suppose β is induced by a homeomorphism $h : D_n \to D_n$. As above we consider the braid β to be embedded in $D \times I$ as the image of $P \times I$ under the homeomorphism $\bar{F} : D \times I \to D \times I$. Recall that $\bar{F}$ is defined by $\bar{F}(x,t) = (F(x,t),t)$ where F is an isotopy of $\bar{h}$ with 1_D.

Let α be the path in $D \times I - \beta$ defined by $\alpha(t) = (d_0, t)$. We specify $(d_0, 0)$ to be the basepoint of $D \times I$. Let $j^0, j^1 : D_n \to D \times I - \beta$ be the maps taking D_n to $D \times \{0\} - P \times \{0\}$ and $D \times \{1\} - P \times \{1\}$, respectively. We let $x_i^0 = j^0 x_i$ and $x_i^1 = \alpha(j^1 x_i)\alpha^{-1}$ where $x_1, x_2, \ldots, x_n$ are the free generators of $F_n = \pi_1(D_n, d_0)$ as in Figure 2.3. Consider the curve $w_i^1 x_{\mu_i}^1 (w_i^1)^{-1}$ in $D \times I - \beta$ where w_i^1 is obtained from w_i by replacing x_i with x_i^1. We show that the isotopy F induces a homotopy of x_i^0 with $w_i^1 x_{\mu_i}^1 (w_i^1)^{-1}$.

Viewing x_i as a map $x_i : (I, \{0,1\}) \to (D, d_0)$ take the composition $\bar{F}(x_i \times 1_I) : I \times I \to D_n \times I$. At $s = 0$ this map is x_i^0. At $s = 1$ the map is $(\bar{h}x_i, 1) = j^1 h x_i$. Hence, this gives us a homotopy $\gamma_s(t) : I \times I \to D_n \times I$ with $\gamma_0 = x_i^0$ and $\gamma_1 = j^1 h x_i = j^1(w_i x_{\mu_i}(w_i)^{-1})$. However, this homotopy moves the base point along the path α. Hence, taking the composition $\alpha(st)\gamma_s(t)\alpha^{-1}(st)$ we have the desired homotopy from x_i^0 to $\alpha\big(j^1(w_i x_{\mu_i}(w_i)^{-1})\big)\alpha^{-1} = w_i^1 x_{\mu_i}^1 (w_i^1)^{-1}$.

2.4 Links

A *link* L is the union of $k \geq 1$ disjoint circles embedded in $\S^3$. A link with $k = 1$ is called a *knot*.

There is a close connection between braids and links. In fact, one of the major reasons why braid theory is so extensively studied is because of its applications to the theory of knots and links. We give one such application in this section. Another will be given in Chapter 4.

Any braid $\beta \in B_n$ can be closed off to give a link $\hat{\beta}$ (see Figure 2.5). Moreover, as the next theorem shows, the action of the braid β on the group F_n gives a presentation of the knot group $\pi_1(S^3 - \hat{\beta})$.

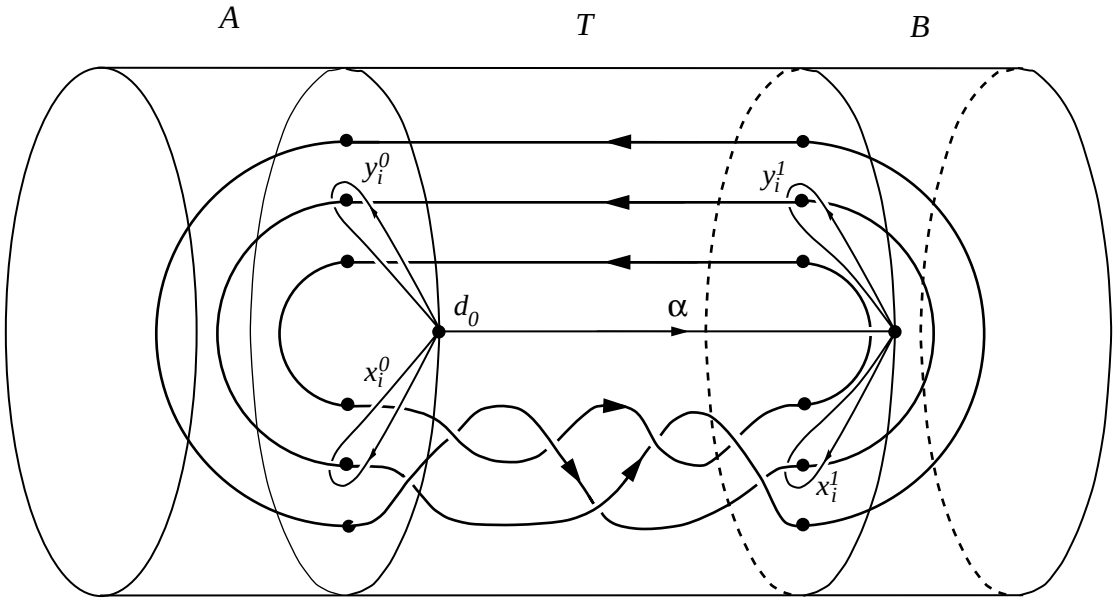


Figure 2.5: The complement in S of a closed braid.

Theorem 5. *Let $\beta \in B_n$ be a braid with β acting on F_n as in* (2.6) – (2.7)*. Then $\pi_1(S^3 - \hat{\beta})$ admits a presentation with generators $x_1, x_2, \ldots, x_n$ and relations given by*

$$x_i = w_i x_{\mu_i} w_i^{-1} \qquad i = 1, 2, \ldots, n \tag{2.8}$$

Proof. Let $\beta \in \mathrm{Aut}(F_n)$ be induced by a homeomorphism $h : D_n \to D_n$. As in section 2.3, we consider the braid β to be embedded in $D \times I$ as the image of $P \times I$ under the homeomorphism $\bar{F} : D \times I \to D \times I$.

Let $S \cong D \times I$ be a closed cylinder containing $\hat{\beta}$ in its interior. Looking at Figure 2.5 we see that the space $S - \hat{\beta}$ can be viewed as having three components. There are the caps A and B at either end, which are both homotopy equivalent to D_n and there is the middle part T, which is the part in which all the braiding occurs. Furthermore, if we cut T down the middle, separating the braided strands from the unbraided

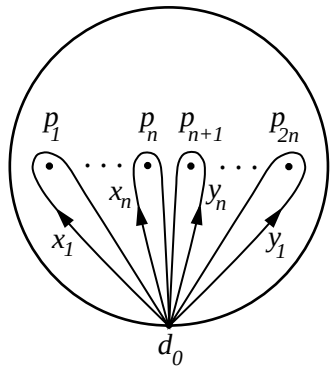


Figure 2.6: Loops in D_{2n} which generate its fundamental group

strands, we obtain a copy of $D \times I - \beta$. We consider $D \times I - \beta$ as embedded in $S - \hat{\beta}$ in this way.

There is a copy of D_{2n} at either end of T. Let $j^0, j^1 : D_{2n} \to S$ be maps identifying D_{2n} with these two copies. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the loops pictured in Figure 2.6 (take note of how the y_i's are labeled and oriented). As above we set $x_i^0 = j^0 x_i$, $x_i^1 = \alpha(j^1 x_i)\alpha^{-1}$, $y_i^0 = j^0 y_i$, and $y_i^1 = \alpha(j^1 y_i)\alpha^{-1}$. The discussion after Theorem 4 shows that x_i^0 is homotopic to $w_i^1 x_{\mu_i}^1 (w_i^1)^{-1}$ inside T.

Let $U = T \cup B$ so that $S = A \cup U$. Then $\pi_1(A)$ is freely generated by $x_1^0, \ldots, x_n^0$ and $\pi_1(U)$ is freely generated by $x_1^1, \ldots, x_n^1$. The fundamental group of the intersection $A \cap U$ is freely generated by $x_1^0, \ldots, x_n^0$ and $y_1^0, \ldots, y_n^0$. The inclusion $A \cap U \subseteq A$ induces a map on the fundamental groups given by $x_i^0, y_i^0 \mapsto x_i^0$ for each $i = 1, 2, \ldots, n$. Similarly, $A \cap U \subseteq U$ induces a map on the fundamental groups given by $x_i^0 \mapsto w_i^1 x_{\mu_i}^1 (w_i^1)^{-1}$ and $y_i^0 \mapsto x_i^1$. From the Seifert–Van Kampen theorem $\pi_1(S - \hat{\beta})$ has the presentation we seek. Hence the theorem follows since $\pi_1(S - \hat{\beta}) \cong \pi_1(S^3 - \hat{\beta})$. $\square$

CHAPTER 3

REPRESENTATIONS OF B_n FROM HOMOLOGY

In this chapter we allow B_n to act on certain topological spaces. Passing to homology will then give the Burau and Krammer representations of the braid group. Explicit matrices for these representations will be given.

3.1 The Burau Representation

Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disk in the complex plane. Let $P = \{p_1, p_2, \ldots, p_n\}$ be n discrete points in D and let $D_n = D - P$. For any loop α in D_n based at d_0 there is a unique integer $\phi\alpha$ called the *total winding number* for α which counts the number of times α winds around the punctures $p_1, p_2, \ldots, p_n$. To be more specific, if a loop α is given by $\Pi_{k=1}^{m} x_{i_k}^{\epsilon_k}$, then $\phi\alpha = \Sigma_{k=1}^{m} \epsilon_k$. This gives a homomorphism $\phi : \pi_1(D_n) \to \mathbb{Z}$.

Let $\tilde{D}_n$ be the regular covering space of D_n corresponding to the kernel of ϕ. $\mathbb{Z} = \langle t \rangle$ acts on $\tilde{D}_n$ as a group of deck transformations. Proposition 6 will show that $H_1(\tilde{D}_n)$ is a free $\mathbb{Z}[t, t^{-1}]$ module of rank $n - 1$.

Let $\beta \in B_n$ be induced by a self-homeomorphism h of D_n. That is, $\beta = h_*$. Then for any loop γ in D_n we have $\phi(h\gamma) = \phi\gamma$ so that h will lift to a homeomorphism $\tilde{h}$ of $\tilde{D}_n$ to itself. Passing to homology gives us an automorphism of $H_1(\tilde{D}_n)$. If h' is any other self-homeomorphism with $h'_* = \beta$ then $h_*^{-1} h'_* = 1$ so that since $\pi_1(\tilde{D}_n) \to \pi_1(D_n)$ is injective we have $\tilde{h}_*^{-1} \tilde{h}'_* = 1$ as a map on $\pi_1(\tilde{D}_n)$. Passing to homology gives $\tilde{h}'_* - \tilde{h}_* = 0$. Hence, the homomorphism $\psi_{\mathrm{r}} : B_n \to \mathrm{GL}(H_1(\tilde{D}_n))$ given

by $\psi_{\mathrm{r}} : \beta \mapsto \tilde{h}_*$ is well defined. It is called the *reduced Burau representation* of B_n. It is said to be reduced because there is a nice n-dimensional representation of which the reduced Burau representation is an irreducilble summand. This n-dimensional representation, which we will also give a construction for, is what is usually called the (unreduced) Burau representation. [1]

Before we give the matrices for the reduced representation we need to first verify the following proposition.

Proposition 6. *$H_1(\tilde{D}_n)$ is a free module of rank $n-1$ over $\mathbb{Z}[t, t^{-1}]$.*

To aid us in the proof of this proposition we first give a geometric description of $\tilde{D}_n$ which will also be of importance later when we examine the connection between the Burau representation and the Alexander polynomial.

Enlarge the punctures of D_n slightly so that we are no longer removing points from D but rather small open disks. Call this new space X. Draw arcs $A_1, A_2, \ldots, A_n$ from the centers of the removed disks to the boundary of X. Cut X along these arcs so that we have two disjoint copies A_i^+ and A_i^- of each arc A_i. Call this space X^*. Let $h_i : A_i^+ \to A_i^-$ be a homeomorphism. Take countably many copies X_j^* of X^*. For each j let $g_j : X_j^* \to X^*$ be a homeomorphism. The space $\tilde{X}$ is defined by taking the disjoint union of the X_j^*'s and identifying $A_i^+ \subseteq X_j^*$ with $A_i^- \subseteq X_{j+1}^*$ via $g_{j+1}^{-1} h_i g_j$. See Figure 3.1.

There is a natural $\mathbb{Z} = \langle t \rangle$ action on $\tilde{X}$ given by $X_j^* \ni x \mapsto g_{j+1}^{-1} g_j x$. This action can be thought of as shifting every point in $\tilde{X}$ one level upward. The orbit space $\tilde{X}/\mathbb{Z}$ of this action is precisely X. Let K be the kernel of the map $\pi_1(X) \to \mathbb{Z}$ which takes

[1] Actually, the reduced representation which we have described here via homology is the transpose of what is usually refered to as the reduced Burau representation, for instance in [4] and [6]. We will not worry ourselves about this, however, since it follow by the braid relations that the transpose of any representation of B_n is also a representation. See [19] for the differences between the usual Burau representation and the one we have given here as well as a two variable representation which generalizes both.

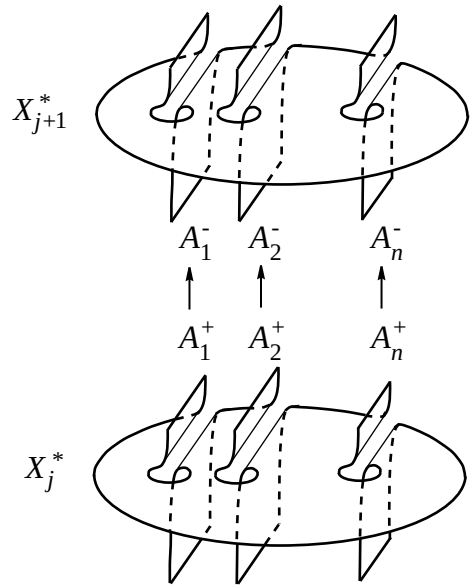


Figure 3.1: Two leaves of the infinite cyclic covering of X.

a loop to its total winding number. Evidently the space $\tilde{X}$ is the regular covering space of X coresponding to K.

Proof of Proposition 6. From the construction above it is clear that the inclusion $X \hookrightarrow D_n$ is a homotopy equivalence which induces an isomorphism of K onto $\ker\phi$. Thus, the inclusion $X \hookrightarrow D_n$ lifts to a map $\tilde{X} \to \tilde{D}_n$ which induces an isomorphism of fundamental groups and, thereby, of homology. Choosing a generator $t \in \tilde{X}/K \cong \tilde{D}_n/\ker\phi$, the induced map on homology becomes a $\mathbb{Z}[t, t^{-1}]$-module isomorphism. Hence, we need only to calculate the homology of X.

Let $A = \bigcup_{j\in\mathbb{Z}} X^*_{2j}$ and $B = \bigcup_{j\in\mathbb{Z}} X^*_{2j+1}$ as subspaces of $\tilde{X}$. The Mayer-Vietoris long exact sequence gives us the following:

$$0 \longrightarrow H_1(\tilde{X}) \xrightarrow{\delta} H_0(A \cap B) \xrightarrow{\gamma} H_0(A) \oplus H_0(B)$$

where the zero on the left comes from the fact that both A and B are homotopy

equivalent to a discrete space. Hence, $H_1(\tilde{X}) \cong \ker \gamma$. Now $H_0(A \cap B)$ is isomorphic to $\bigoplus_{i \in \mathbb{Z}} \mathbb{Z}^n$ since for each j the intersection $X_j^* \cap X_{j+1}^*$ is homotopy equivalent to a space of n discrete points. Let $\{a_{j,1}, a_{j,2}, \ldots, a_{j,n}\}_{j \in \mathbb{Z}}$ be a $\mathbb{Z}$-basis for $H_0(A \cap B)$. It is easily seen that $\{a_{j,1} - a_{j,2}, \ldots, a_{j,n-1} - a_{j,n}\}_{j \in \mathbb{Z}}$ is a basis for $\ker \gamma$. Let $\tilde{d}_0 \in \tilde{X}$ be a lift of the basepoint $d_0 \in X$ (say $\tilde{d}_0 \in X_0^*$). Denote by v_i the element of $H_1(\tilde{X})$ represented by the lift of the loop $x_i x_{i+1}^{-1}$. We have $\delta v_i = a_{0,i} - a_{0,i+1}$ and for all $j \in \mathbb{Z}$ we have $\delta(t^j v_i) = a_{j,i} - a_{j,i+1}$. Hence, $\{v_1, v_2, \ldots, v_{n-1}\}$ is a basis for $H_1(\tilde{X})$ as a $\mathbb{Z}[t, t^{-1}]$-module. $\square$

Our statement in the above proof that $\delta v_i = a_{0,i} - a_{0,i+1}$ deserves some elaboration. Let $\tilde{\alpha}$ be a loop in $\tilde{X}$ representing a cycle in $H_1(\tilde{X})$. Project $\tilde{\alpha}$ to a loop α in X. Draw a diagram consisting of horizontal lines (one for each level of the covering space $\tilde{X}$) with the numbers $1, 2, \ldots, n$ marked along the bottom (one for each curve A_i). As α crosses A_i going to the right, the loop $\tilde{\alpha}$ passes from X_j^* to X_{j+1}^* for some j. For each such crossing draw an arrow in the ith column of our diagram going up from the jth to the $(j+1)^{\text{st}}$ line. Similarly, each time α crosses A_i going to the left draw a down arrow. See for instance Figure 3.2. As a quick examination of the definition of the connecting homomorphism δ will show, this diagram tells us the element in $H_0(A \cap B)$ to which $\tilde{\alpha}$ is mapped by δ. Each upward pointing arrow in the ith column from the jth line to the $(j+1)^{\text{st}}$ line represents the element $a_{j,i}$. Similarly, downward pointing lines represent negative elements.

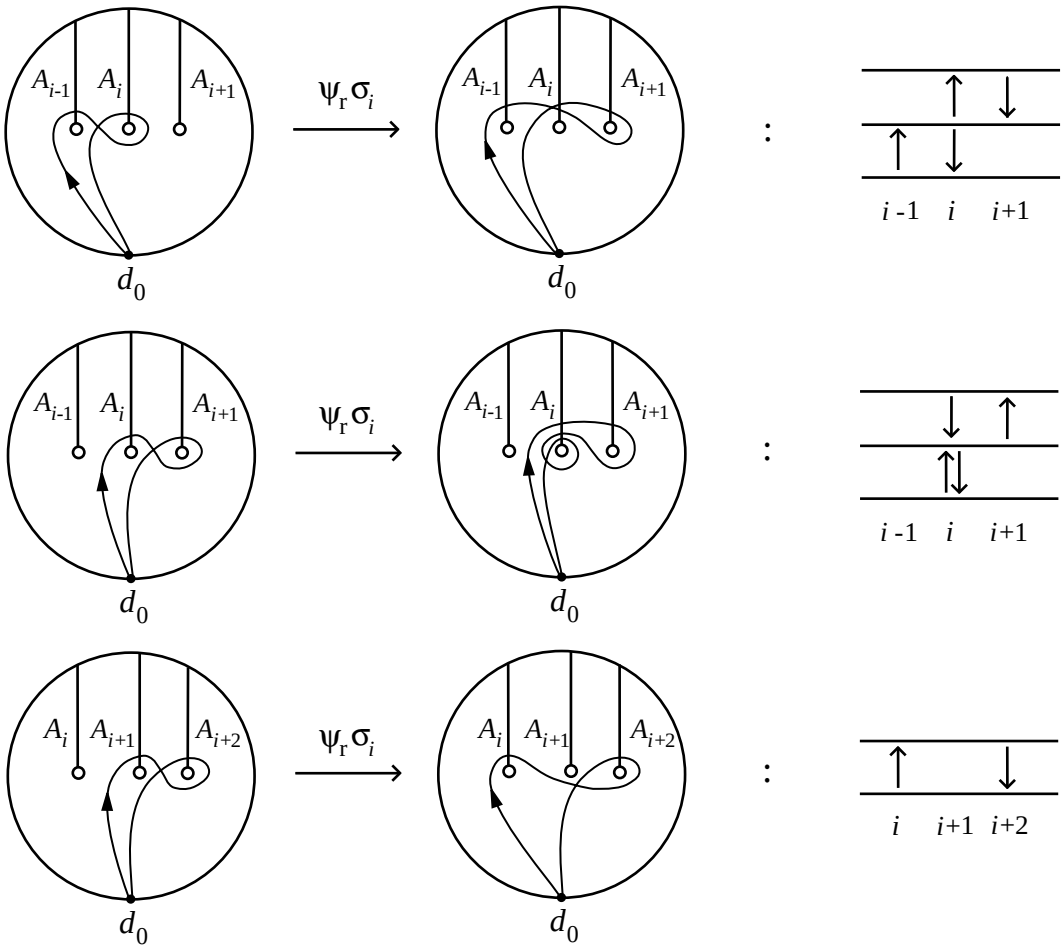


Figure 3.2: Calculating the action of $\psi_r\sigma_i$ on $H_1(\tilde{X})$ using level diagrams.

3.2 Matrices for the Burau Representation

With this last observation of the previous section we can easily obtain matrices for the Burau representation. As Figure 3.2 shows, the action of σ_i on $H_1(\tilde{X})$ is given by

$$\psi_{\mathrm{r}}\sigma_i(v_j) = \begin{cases} v_j + tv_{j+1}, & j = i-1 \\ -tv_j, & j = i \\ v_{j-1} + v_j, & j = i+1 \\ v_j, & \text{otherwise} \end{cases} \tag{3.1}$$

Hence, the matrices for the representation relative to the basis $\{v_1, v_2, \ldots, v_{n-1}\}$ are given by

$$\psi_{\mathrm{r}}\sigma_1 = \begin{bmatrix} -t & 1 & \\ 0 & 1 & \\ & & I \end{bmatrix}, \quad \psi_{\mathrm{r}}\sigma_i = \begin{bmatrix} I & & & & \\ & 1 & 0 & 0 & \\ & t & -t & 1 & \\ & 0 & 0 & 1 & \\ & & & & I \end{bmatrix}, \quad \psi_{\mathrm{r}}\sigma_{n-1} = \begin{bmatrix} I & & \\ & 1 & 0 \\ & t & -t \end{bmatrix} \tag{3.2}$$

It can be checked directly that the matrices $\psi_{\mathrm{r}}\sigma_i$ satisfy the braid relations (2.1)-(2.2).

As mentioned above, the reduced Burau representation is a summand of an n-dimensional representation called the unreduced Burau representation. This unreduced representation arises in much the same way as the reduced representation. Consider $D_{n+1} \subseteq D_n$ obtained by removing a point of D_n. We have $B_n \subseteq \mathrm{Aut}(F_{n+1})$ in an obvious way, namely, β acts on x_i by (2.6) and fixes x_{n+1}. Hence, we get an action of B_n on the regular covering space $\tilde{D}_{n+1}$ of D_{n+1} corresponding to the kernel of the total winding number map. Arguments similar to those used in the proof of Proposition 6 show that the first homology of $\tilde{D}_{n+1}$ is a $\mathbb{Z}[t, t^{-1}]$-module with basis $\{u_1, u_2, \ldots, u_n\}$ where u_i is a lift of $x_i x_{n+1}^{-1}$. By passing the action of B_n on $\tilde{D}_{n+1}$ to homology we obtain the *unreduced Burau representation* $\psi : B_n \to \mathrm{GL}(H_1(\tilde{D}_{n+1})) \cong \mathrm{GL}(n, \mathbb{Z}[t, t^{-1}])$.

A calculation similar to the one carried out in Figure 3.2 will give the action of σ_i on $H_1(\tilde{D}_{n+1})$:

$$\psi\sigma_i u_j = \begin{cases} (1-t)u_j + tu_{j+1}, & j = i \\ u_{j+1}, & j = i+1 \\ u_j, & \text{otherwise} \end{cases} \tag{3.3}$$

Hence, the matrices for the representation relative to the basis $\{u_1, u_2, \ldots, u_n\}$ are given by

$$\psi\sigma_i = \begin{bmatrix} I & & & \\ & 1-t & 1 & \\ & t & 0 & \\ & & & I \end{bmatrix} \tag{3.4}$$

where the term $1-t$ occurs in the ith row and column.

Consider the collection $\{u'_1, u'_2, \ldots, u'_n\}$ where $u'_n = u_n$ and $u'_i = u_i - u_{i+1}$ for $i < n$. This collection is another basis for $H_1(\tilde{D}_{n+1})$. The braid action relative to this basis can be calculated using (3.3) as follows:

$$\begin{aligned} \psi\sigma_i(u'_{i-1}) = \psi\sigma_i(u_{i-1} - u_i) &= u_{i-1} - ((1-t)u_i + tu_{i+1}) \\ &= u_{i-1} - u_i + t(u_i - u_{i+1}) = u'_{i-1} + tu'_i \end{aligned} \tag{3.5}$$

$$\begin{aligned} \psi\sigma_i(u'_i) = \psi\sigma_i(u_i - u_{i+1}) &= (1-t)u_i + tu_{i+1} - u_i \\ &= tu_{i+1} - tu_i = -tu'_i \end{aligned} \tag{3.6}$$

$$\begin{aligned} \psi\sigma_i(u'_{i+1}) = \psi\sigma_i(u_{i+1} - u_{i+2}) &= u_i - u_{i+2} \\ &= u_i - u_{i+1} + u_{i+1} - u_{i+2} = u'_i + u'_{i+1} \end{aligned} \tag{3.7}$$

$$\psi\sigma_i(u'_j) = u'_j \qquad for j \neq i-1, i, i+1 \tag{3.8}$$

Notice that we have $\psi\sigma_{n-1}{u_n}' = u'_{n-1} + u'_n$ even though the calculation in (3.7) does not apply in this case. Indeed, $\psi\sigma_{n-1}{u_n}' = \psi\sigma_{n-1}u_n = u_{n-1} = u_{n-1} - u_n + u_n = u'_{n-1} + u'_n$.

Comparing these equations with (3.1) we see that for any braid $\beta \in B_n$ we have

$$\psi\beta = \left[\begin{array}{ccc|c} & & & * \\ & \psi_{\mathrm{r}}\beta & & \vdots \\ & & & * \\ \hline 0 & \cdots & 0 & 1 \end{array}\right] \tag{3.9}$$

so that the reduced Burau representation is a summand of the unreduced Burau representation.

Consider the $(n-1)\times n$ matrix M_β obtained by deleting the nth row of the Burau matrix $\psi\beta$ in (3.9). To aid us in future calculations we define a map $v : B_n \to \bigoplus_{i=1}^{n-1} \mathbb{Z}[t, t^{-1}]$ which takes a braid $\beta \in B_n$ to the nth column of the matrix M_β. This allows us to write

$$\psi\beta = \left[\begin{array}{c|c} \psi_{\mathrm{r}}\beta & v(\beta) \\ \hline 0 & 1 \end{array}\right] \tag{3.10}$$

Note that v is not a homomorphism. Instead we have

$$v(\beta_1\beta_2) = v\beta_1 + (\psi_{\mathrm{r}}\beta_1)(v\beta_2) \tag{3.11}$$

which follows by virtue of the fact that ψ and ψ_{r} *are* homomorphisms. Also, (3.5) - (3.8) give the following:

$$v(\sigma_i) = \begin{cases} 1, & i = n-1 \\ 0, & i < n-1 \end{cases} \tag{3.12}$$

We should remark that the unreduced Burau representation arises by letting B_n act on spaces other than $\tilde{D}_{n+1}$. For instance, consider the $2n$-punctured disk D_{2n}. Let $\pi_1(D_{2n}) = F_{2n}$ be generated by the loops $x_1, \ldots, x_n, y_n, \ldots, y_1$ drawn in Figure 2.6. Notice that the y_i's are oriented differently than the x_i's and are written in decreasing order. Let $\Sigma u_j^{\epsilon_j}$ be any word in the generators of F_{2n}. Define a map $\phi : F_{2n} \to \mathbb{Z}$ by $\phi : \Sigma u_j^{\epsilon_j} \mapsto \Sigma\epsilon_j$. Because of the opposite orientations for the generators of F_{2n} this map is different from the total winding number map as we defined it above. Let $\tilde{D}_{2n}$ be the regular covering space corresponding to $\ker\phi$. Let β in B_n be induced

by $h : D_n \to D_n$. Then the restriction of h to D_{2n} will lift to a homeomorphism of $\tilde{D}_{2n}$ to itself. Passing to homology gives an automorphism of $H_1(\tilde{D}_{2n})$. In this way we obtain a homomorphism $B_n \to H_1(\tilde{D}_{2n})$. We wish to show this homomorphism to be equivalent to the unreduced Burau representation. As we did for the proof of Proposition 6 we first give a geometric realization of the space $\tilde{D}_{2n}$.

Enlarge the punctures of D_{2n} so that we are no longer removing points from D_{2n} but rather small open disks. Call this space Y. Draw arcs $A_1, \ldots, A_n$ in Y where the arc A_i joins the ith and the $(2n-i+1)^{\text{st}}$ punctures. Cut Y along these arcs to obtain Y^*. The space Y^* then has two copies, A_i^+ and A_i^-, of each arc A_i. Define $\tilde{Y}$ to be equal to the disjoint union of countably many copies Y_j^* of Y^* with $A_i^+ \subseteq Y_j^*$ identified with $A_i^- \subseteq Y_{j+1}^*$. $\tilde{Y}$ is a regular covering of Y with a $\mathbb{Z} = \langle t \rangle$ action given by shifting one level upward. From the comments at the beginning of the proof of Proposition 6 we have $H_1(\tilde{D}_{2n}) \cong H_1(\tilde{Y})$ as $\mathbb{Z}[t, t^{-1}]$-modules.

Proposition 7. *The first homology group of the space $\tilde{Y}$ is a free $\mathbb{Z}[t, t^{-1}]$-module of rank $2n-1$ generated by $\nu_1, \ldots, \nu_{n-1}$, $\omega_1, \ldots, \omega_{n-1}$, and ρ where ν_i is a lift of $x_i x_{i+1}^{-1}$, ω_i is a lift of $y_i y_{i-1}^{-1}$, and ρ is a lift of $x_n y_n^{-1}$ (all relative to $\tilde{d}_0$).*

Proof. Divide Y down the middle letting L denote the left side and R the right side. Then both L and R are homeomorphic to the space X^* defined in section 3.1. Hence, the proof of Proposition 6 tells us that $H_1(\tilde{L})$ and $H_1(\tilde{R})$ are both free $Z[t, t^{-1}]$-modules of rank $n-1$ generated by $\nu_i = \widetilde{x_i x_{i+1}^{-1}}$ and $\omega_i = \widetilde{y_i y_{i-1}^{-1}}$, respectively. The intersection $\tilde{L} \cap \tilde{R}$ is homotopy equivalent to the countably infinite discrete space $\coprod_{j \in \mathbb{Z}} \{t^j \tilde{d}_0\}$.

We have the following Mayer-Vietoris sequence:

$$0 \longrightarrow H_1(\tilde{L}) \oplus H_1(\tilde{R}) \longrightarrow H_1(\tilde{Y}) \xrightarrow{\delta} H_0(\tilde{L} \cap \tilde{R}) \xrightarrow{\eta} H_0(\tilde{L}) \oplus H_0(\tilde{R})$$

Hence, if $K = \ker \eta$ then we have $H_1(\tilde{Y}) \cong H_1(\tilde{L}) \oplus H_1(\tilde{R}) \oplus K$. Since $H_0(\tilde{L} \cap \tilde{R}) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z} a_j$, where we have set a_j equal to $t^j \tilde{d}_0$, we see that K is freely generated over

$\mathbb{Z}[t, t^{-1}]$ by $a_0 - a_1$. But a quick examination of the definition of the connecting homomorphism δ shows that $\delta\rho = a_0 - a_1$. This concludes the proof. $\square$

With this result the action of B_n on $H_1(\tilde{D}_{2n})$ can be computed. B_n acts trivially on $y_1, \ldots, y_n$ so that B_n has trivial action on $\omega_1, \ldots, \omega_{n-1}$. The action of B_n on the the elements $\nu_1, \ldots, \nu_{n-1}$ is induced from the braid action on $H_1(\tilde{L})$. This is, by definition, the action of the reduced Burau representation given by (3.1). Lastly, by an explicit calculation we have $\sigma_{n-1} : \rho \mapsto \rho + \nu_{n-1}$ and $\sigma_i : \rho \mapsto \rho$ for $i < n-1$. This exactly corresponds to the action of $\psi\sigma_{n-1}$ on u'_n as given by (3.7). Hence, forgetting about the trivial action on $\omega_1, \ldots, \omega_{n-1}$, we find that the matrices for this representation relative to $\{\nu_1, \ldots, \nu_{n-1}, \rho\}$ are precisely those of the unreduced Burau representation given by (3.10).

3.3 The Krammer Representation

Still using the notation D, $P = \{p_1, \ldots, p_n\}$, $D_n = D - P$ introduced in section 3.1 we construct the space C of all unordered pairs of distinct points in D_n. That is, $C = J/S_2$ where $J = \{(x, y) \in D_n \times D_n \mid x \neq y\}$ and S_2 acts on J by sending (x, y) to (y, x). Points in C are denoted by $\{x, y\}$. The projection $J \to C$, taking (x, y) to $\{x, y\}$, is evidently a two-fold covering map. Let d_0 and d'_0 be points on the boundary of D_n which are very close to each other. We take $c_0 = \{d_0, d'_0\}$ to be the basepoint of C.

Let $\alpha : I \to C$ be a path in C based at c_0. Lift this path to J relative to (d_0, d'_0). This lifted path must be of the form (α_1, α_2) where α_1, $\alpha_2 : I \to D_n$ are paths in D_n. As a matter of notation we may then write $\alpha(s) = \{\alpha_1(s), \alpha_2(s)\}$. Note that if α is a loop then $\{\alpha_1(0), \alpha_2(0)\} = \{\alpha_1(1), \alpha_2(2)\} = \{d_0, d'_0\}$. Hence, the paths α_1 and α_2 in D_n are either both loops or they may be composed with one another. For example, consider the loops γ_1 and γ_2 in Figure 3.3.

If α_1 and α_2 are any paths in D_n with $\alpha_1(s) \neq \alpha_2(s)$ for all s, then (α_1, α_2) is a

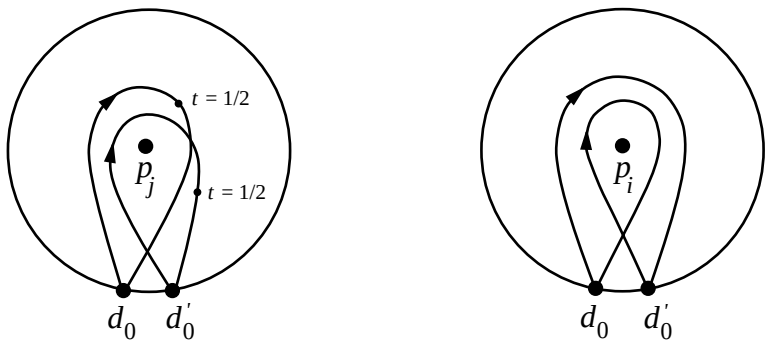


Figure 3.3: Two loops γ_1 and γ_2 in C. The point labeled $s = 1/2$ shows how γ_1 is parametrized.

path in J. The image of this path under $J \to C$ is a path in C denoted by $\{\alpha_1, \alpha_2\}$. If α_1 is the path having constant value u, then we write $\{\alpha_1, \alpha_2\} = \{u, \alpha_2\}$.

Let $\alpha : I \to C$ represent an element of $\pi_1(C)$. Define maps a and b from $\pi_1(C)$ to $\mathbb{Z}$ as follows: If α_1 and α_2 are both closed loops then $a(\alpha)$ is the sum of the winding numbers of α_1 and α_2 around the puncture points $p_1, \ldots, p_n$ where we specify clockwise as the positive orientation. If α_1 and α_2 are not both closed loops then $a(\alpha)$ is the winding number of the loop $\alpha_1\alpha_2$ around the puncture points. The map b is defined by first composing the map $I \ni s \mapsto (\alpha_1(s) - \alpha_2(s))/|\alpha_1(s) - \alpha_2(s)| \in S^1$ with the projection $S^1 \to \mathbb{R}P^1$ to obtain a loop in $\mathbb{R}P^1$. The corresponding element of $H_1(\mathbb{R}P^1) \cong \mathbb{Z}$ is $b(\alpha)$. Hence, a measures how many times the loops α_1 and α_2 wind around the puncture points while b measures how many times they wind around each other.

Let $\langle q, t \rangle$ denote the free Abelian group generated by q and t. Define a map $\phi : \pi_1(C) \to \langle q, t \rangle$ by $\phi : \alpha \to q^{a(\alpha)} t^{-b(\alpha)}$. For example, we have $\phi\gamma_1 = q^2$ and $\phi\gamma_2 = tq^2$ where γ_1 and γ_2 are the loops in Figure 3.3.

Let $\tilde{C} \to C$ be the regular covering of C corresponding to the kernel of ϕ. Choose

a lift $\tilde{c}_0$ of c_0 to $\tilde{C}$. The group $\langle q, t\rangle$ acts on $\tilde{C}$ as a group of deck transformations. Hence, the homology group $H_2(\tilde{C})$ becomes a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module.

Let h be a self-homeomorphism of D_n fixing the boundary. Then h induces a homeomorphism $C \to C$, also denoted by h, which is given by $h : \{x, y\} \mapsto \{h(x), h(y)\}$. This map h lifts to a homeomorphism $\tilde{h} : \tilde{C} \to \tilde{C}$ which commutes with the covering transformations q and t. Hence, $\tilde{h}$ induces a $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module isomorphism $\tilde{h}_* : H_2(\tilde{C}) \to H_2(\tilde{C})$.

Definition 8. *The representation $\kappa : B_n \to Aut(H_2(\tilde{C}))$ induced by $h \mapsto \tilde{h}_*$ is the called the* Krammer representation.

3.4 Forks

We would like to have a nice way of representing elements of the homology module $H_2(\tilde{C})$. As we will see, one way of doing this is with forks.

A *fork* F in D_n is an embedded tree $F \subseteq D$ formed by three edges and four vertices d_0, p_i, p_j, and z such that $F \cap \partial D = \{d_0\}$, $F \cap P = \{p_i, p_j\}$, and z is a common vertex for all three edges. The edge H connecting d_0 to z is called the *handle* of the fork. The other two edges, which connect z to p_i and p_j, respectively, are called the *tines* of the fork. The union T of both of the tines of F is an arc in D with endpoints p_i and p_j. This arc T is usually called the *tine edge* of the fork F.

In a small neighborhood of z the handle H lies on one side of T. We orient T so that this distinguished side lies on its right. The handle H also has a distinguished side determined by d_0'. Simultaneously pushing T and H to their distinguished sides, leaving p_i and p_j fixed and pushing d_0 to d_0', gives a *parallel copy* $F' = T' \cup H'$ of F (see Figure 3.4). We have $T \cap T' = \{p_i, p_j\}$, $H \cap H' = \emptyset$, and $F' \cap \partial D = \{d_0'\}$. This parallel fork F' has an orientation which is induced from that of F.

For any fork F we define a surface $\tilde{\Sigma}_F$ in $\tilde{C}$ as follows. Set $\Sigma_F = \{\{x, y\} \in C \mid x \in T - P \text{ and } y \in T' - P\}$. Clearly, Σ_F is homeomorphic to an open square. Let α_1

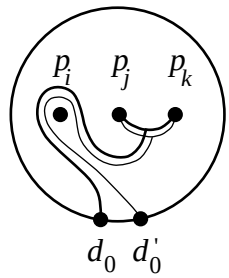


Figure 3.4: A fork F and its parallel copy F'

be a path from d_0 to z along H and let α_2 be a path from d_0' to z' along H'. Consider the path $\alpha = \{\alpha_1, \alpha_2\}$ in C joining c_0 to $\{z, z'\}$. Lift α to a path $\tilde{\alpha}$ in $\tilde{C}$ relative to $\tilde{c}_0$. We define $\tilde{\Sigma}_F$ to be the lift of Σ_F to $\tilde{C}$ which contains $\tilde{\alpha}(1)$. The surfaces Σ_F and $\tilde{\Sigma}_F$ have natural orientations which are determined by the orientations of T and T'.

For each $i = 1, \ldots, n$ let $U_i \subseteq D_n$ be an ϵ-neighborhood about the ith puncture. Let U be the set of points $\{x, y\} \in C$ such that at least one of x or y lies in $\bigcup_{i=1}^n U_i$. Let $\tilde{U}$ be the pre-image of U under the covering map $\tilde{C} \to C$. If F is any fork, then $\tilde{\Sigma}_F$ is an open square which has a closed subsquare $S_F \subseteq \tilde{\Sigma}_F$ such that $\tilde{\Sigma}_F - S_F \subseteq \tilde{U}$. Hence, S_F represents a relative homology class $[S_F] \in H_2(\tilde{C}, \tilde{U})$. A direct calculation in [3] shows that $(q-1)^2(qt+1)[S_F]$ maps to 0 under the boundary homomorphism $\partial : H_2(\tilde{C}, \tilde{U}) \to H_1(\tilde{U})$. Hence, $(q-1)^2(qt+1)[S_F] = j_* v_F$ where j_* is the map $H_2(\tilde{C}) \to H_2(\tilde{C}, \tilde{U})$ induced by inclusion and v_F is a homology class is $H_2(\tilde{C})$. Hence, associated with any fork F is a 2-dimensional homology class v_F. This homology class is represented by an immersed closed surface which is identical to $(q-1)^2(qt+1)\tilde{\Sigma}_F$ outside an ϵ-neighborhood of the puncture points. It follow by calculations performed in [3] that the homology class v_F is well defined by the fork F.

Let $F_{i,j}$ be the fork lying entirely in the closed lower half of D such that its tine edge has endpoints p_i and p_j. Such a fork $F_{i,j}$ for $1 \leq i < j \leq n$ is called a *standard fork* and is determined uniquely up to isotopy by the points p_i and p_j. Let $v_{i,j}$ be the

homology class in $H_2(\tilde{C})$ determined by $F_{i,j}$. It is shown in [3] that $H_2(\tilde{C})$ is a free $\mathbb{Z}[q^{\pm 1}, t^{\pm 1}]$-module with basis $\{v_{i,j}\}_{1 \leq i < j \leq n}$. In fact, this result was essentially proven by Lawrence in [15].

3.5 Matrices for the Krammer Representation

We now concentrate on giving explicit formulas for the action of B_n on $H_2(\tilde{C})$. We accomplish this in a geometric way by viewing $\tilde{C}$ as a path space. Namely, $\tilde{C}$ is the space of all paths $\alpha : I \to C$ with $\alpha(0) = c_0$ where we identify α with β if $\alpha\beta^{-1}$ is in $\ker \phi$. The map $\tilde{C} \to C$ is given by $\alpha \mapsto \alpha(1)$. The base point of $\tilde{C}$ is the map having constant value c_0. The group $\pi_1(C)/\ker \phi = \langle q, t \rangle$ acts on $\tilde{C}$ by composition. That is, if γ is a loop in C, then the map $\alpha \mapsto \gamma\alpha$ defined by composition with γ is a self-homeomorphism of $\tilde{C}$. Moreover, this action of $\pi_1(C)$ on $\tilde{C}$ descends to $\pi_1(C)/\ker \phi$ since $\gamma \in \ker \phi \Rightarrow \gamma\alpha\alpha^{-1} \in \ker \phi \Rightarrow \gamma\alpha = \alpha$. Hence, the action of $\langle q, t \rangle$ on $\tilde{C}$ is given by $\gamma\alpha = \phi(\gamma)\alpha$.

Let F_1, F_2, and F_3 be the forks pictured in Figure 3.5. Thinking of each fork F as standing for its associated 2-dimensional homology class v_F, the following lemma shows how to write these forks in terms of the standard forks.

Lemma 9. *The forks F_1, F_2, and F_3 pictured in Figure 3.5 can be expressed in terms of standard forks as follows (cf. [13] pp. 463-464)*

$$F_1 = q^2 F_{i,i+1} \tag{3.13}$$

$$F_2 = -tq^2 F_{i,i+1} \tag{3.14}$$

$$F_3 = (1-q)F_{i-1,i} + (q^2 - q)F_{i,i+1} + qF_{i-1,i+1} \tag{3.15}$$

Proof. Notice that F_1 and $F_{i,i+1}$ have the same tine edge so that $\Sigma_{F_1} = \Sigma_{F_{i,i+1}}$. Hence F_1 and $F_{i,i+1}$ represent the same homology class up to an application of a covering transformation. Recall that points in $\tilde{\Sigma}_{F_1}$ are paths in C joining c_0 to Σ_{F_1} along the

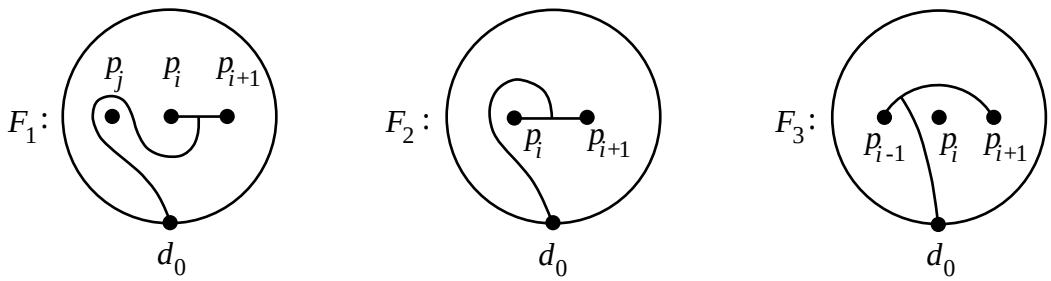


Figure 3.5: The nonstandard forks F_1, F_2, and F_3

handle of the fork. With this observation it is clear that $\gamma_1\tilde{\Sigma}_{F_{i,i+1}} = \tilde{\Sigma}_{F_1}$ where γ_1 is pictured in Figure 3.3. Since $\phi\gamma_1 = q^2$ equation (3.13) follows.

In the case of F_2, notice that F_2 and $F_{i,i+1}$ also have the same tine edge so that Σ_{F_2} and $\Sigma_{F_{i,i+1}}$ are set-wise equal to each other. However, the handles of these forks approach the tines from different sides. Hence Σ_{F_2} and $\Sigma_{F_{i,i+1}}$, and thereby $\tilde{\Sigma}_{F_2}$ and $\tilde{\Sigma}_{F_{i,i+1}}$, have different orientations. Thus, F_2 and $F_{i,i+1}$ represent the same homology class up to an application of a covering transformation and a change of orientation. It is clear that $\gamma_2\tilde{\Sigma}_{F_{i,i+1}} = \tilde{\Sigma}_{F_2}$ as sets where γ_2 is pictured in Figure 3.3. Hence, as $\phi\gamma_2 = tq^2$, we have $F_2 = -tq^2F_{i,i+1}$ where the minus sign accounts for the change in orientation. This establishes equation (3.14). Next we prove equation (3.15).

As above, let $U \subseteq C$ be an ϵ-neighborhood of the puncture points. Let $S_{F_{i-1,i+1}} \subseteq \tilde{\Sigma}_{F_{i-1,i+1}}$ be a closed subsquare of the open square $\tilde{\Sigma}_{F_{i-1,i+1}}$ representing a relative homology class $[S_{F_{i-1,i+1}}] \in H_2(\tilde{C}, \tilde{U})$. We may deform $F_{i-1,i+1}$ so that a small part of its tine edge, and also the tine edge of its parallel copy $F'_{i-1,i+1}$, lies in U. This allows us to write $[S_{F_{i-1,i+1}}] = [S_1] + [S_2] + [S_3] + [S_4]$ where the S_j's are subsquares of $S_{F_{i-1,i+1}}$ labeled by quadrant according to Figure 3.6. We do the same thing to F_3. Hence we obtain $[S_{F_3}] = [S'_1] + [S'_2] + [S'_3] + [S'_4]$ as relative homology classes where $S_{F_3} \subseteq \tilde{\Sigma}_{F_3}$ is a closed subsquare of $\tilde{\Sigma}_{F_3}$ with boundary in $\tilde{U}$.

The images of S_j and S'_j under the covering map $\tilde{C} \to C$ are identical for each

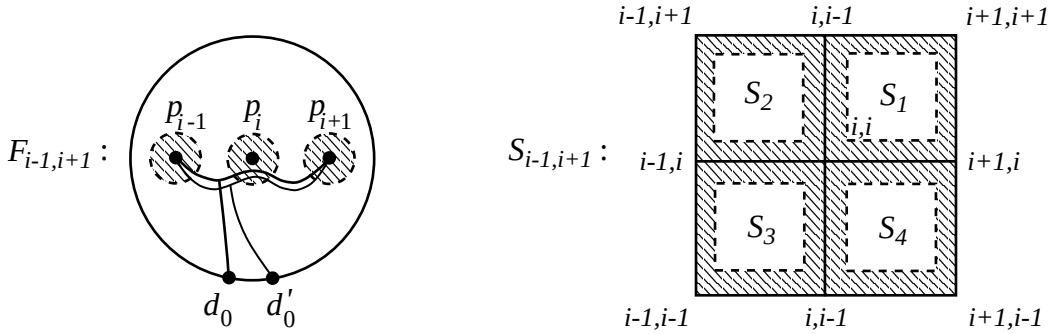


Figure 3.6: Deforming $F_{i-1,i+1}$ so that a small part of its tine edge, and also the tine edge of its parallel copy, lies in U. This allows us to write $[S_{F_{i-1,i+1}}] = [S_1] + [S_2] + [S_3] + [S_4]$.

$j = 1, 2, 3, 4$. Hence, S_j is equal to S'_j for each $j = 1, 2, 3, 4$ up to an application of a covering transformation. The precise relations between these surfaces are given by

$$S'_1 = q^2 S_1 \tag{3.16}$$

$$S'_3 = S_3 \tag{3.17}$$

$$S'_2 = qS_2 \tag{3.18}$$

$$S'_4 = qS_4 \tag{3.19}$$

All one has to do to establish this is to find the appropriate loop γ in C to multiply by. For the first equation, the loop in question is the γ_1-like loop (see Figure 3.3) which winds around the ith puncture. For the last two equations the loops are given by $\{d_0, \delta'_0\}$ and $\{\delta_0, d'_0\}$, respectively. Here δ_0 and δ'_0 are loops in D_n based at d_0 and d'_0, respectively, which wind around p_i in the clockwise direction.

Hence, we have

$$\begin{aligned} [S_{F_3}] &= [S'_1] + [S'_2] + [S'_3] + [S'_4] \\ &= q^2[S_1] + q[S_2] + [S_3] + q[S_4] \end{aligned} \tag{3.20}$$

so that

$$[S_{F_3}] - q[S_{F_{i-1,i+1}}] = (q^2 - q)[S_1] + (1 - q)[S_3]. \tag{3.21}$$

Now, $S_1 = S_{F_{i,i+1}}$ and $S_3 = S_{F_{i-1,i}}$. Hence, multiplying equation (3.21) by $(q - 1)^2(qt + 1)$ we deduce (3.15). □

There are obvious analogues of lemma 9 for forks with handles different from those of F_1, F_2, and F_3. For instance, we have

$$= q^{-2}F_{i,i+1} \qquad \text{and} \qquad = -t^{-1}q^{-2}F_{i,i+1}. \tag{3.22}$$

Now that we have proved Lemma 6 we may give the matrices for the Krammer representation.

Theorem 10. *Let $\{j, k\} \cap \{i, i + 1\} = \emptyset$. The action of $\kappa\sigma_i$ on $H_1(\tilde{C})$ is given as follows (cf. [22] page 14):*

$$(\kappa\sigma_i)v_{j,k} = v_{j,k} \tag{3.23}$$

$$(\kappa\sigma_i)v_{i+1,j} = v_{i,j} \tag{3.24}$$

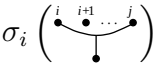
$$(\kappa\sigma_i)v_{j,i+1} = v_{j,i} \tag{3.25}$$

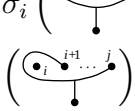
$$(\kappa\sigma_i)v_{i,j} = -tq(q - 1)v_{i,i+1} + (1 - q)v_{i,j} + qv_{i+1,j} \tag{3.26}$$

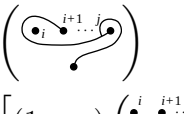
$$(\kappa\sigma_i)v_{i,i+1} = -tq^2v_{i,i+1} \tag{3.27}$$

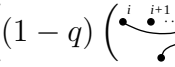
$$(\kappa\sigma_i)v_{j,i} = (1 - q)v_{j,i} + qv_{j,i+1} + q(q - 1)v_{i,i+1} \tag{3.28}$$

Proof. Equations (3.23)–(3.25) are obvious. Equations (3.27) and (3.28) follow directly from (3.14) and (3.15), respectively. Equation (3.26) requires a calculation:

$\sigma_i F_{i,j} = \sigma_i$  (3.29)

$=$  (3.30)

$= -tq^2$  (3.31)

$= -tq^2 \Big[(1 - q)$  $+ (q^2 - q)$ $+ q$ $\Big]$ (3.32)

$$= (1 - q)F_{i,j} - t(q^2 - q)F_{i,i+1} + qF_{i+1,j} \tag{3.33}$$

The first and second equalities are by definition; the third follows from (3.22); the fourth follows from (3.15); and the last equality follows from (3.22). $\square$

CHAPTER 4
THE ALEXANDER POLYNOMIAL

The Alexander polynomial is a link invariant first discovered by J.W. Alexander in 1928. The Burau representation is closely related to the Alexander polynomial as the following theorem shows.

Theorem 11. *Let $\beta \in B_n$ be a braid on n strands. Then*

$$(1 + t + \cdots + t^{n-1})\Delta_{\hat{\beta}}(t) = \det(\psi_r\beta - I) \tag{4.1}$$

where $\Delta_{\hat{\beta}}(t)$ is the reduced Alexander polynomial of the link $\hat{\beta}$.

The purpose of this section is to give a proof of Theorem 11 which does not require the use of the free differential calculus developed by Fox. Rather, we will give a proof which is more in line with the manner in which we obtained the Burau representation. In the first part we briefly give the definition of the Alexander polynomial. The second part then deals with the geometric aspect of Theorem 11. Finally, in the third part we finish off the theorem with a bit of algebra.

4.1 Definition of $\Delta_L(t)$

What follows is a quick definition of the reduced Alexander polynomial of a link. For a more thorough treatment see [16], [6], or just about any other book on knot theory.

Let L be a link in S^3. Let Σ be a Seifert Surface for the link L. That is, Σ is a compact connected orientable 2-manifold with $\partial\Sigma = L$. Let N be a regular neighborhood of the link L. That is, N is a disjoint union of solid tori $D \times S^1$

embedded in S^3 so that the link L is the image under the embedding of the center meridians $\{0\} \times S^1$. Let C be the closure of $S^3 - N$. Cut C along Σ so that we have two disjoint copies Σ^+ and Σ^- of the surface Σ. Call this space C^*. Now, as we did in our construction of the infinite cyclic cover $\tilde{X}$, we take countably many copies C_j^* of C^* and identify $\Sigma^+ \subseteq C_j^*$ with $\Sigma^- \subseteq C_{j+1}^*$. We denote the resulting space by C^∞. Again, as with $\tilde{X}$, there is a natural $\mathbb{Z} = \langle t \rangle$ action on C^∞ which is defined by "shifting one level upward." In fact, our construction of C^∞ parallels that of $\tilde{X}$ almost exactly; for, as is evident, C^∞ is a regular covering space of C corresponding to the kernel of the map $\phi : \pi_1(C) \to \mathbb{Z}$ which takes a curve in C to its algebraic intersection number with the surface $\Sigma \subseteq C$.

We will see in section 4.2 that $H_1(C^\infty)$ is *finitely presented* as a $\mathbb{Z}[t, t^{-1}]$-module. That is, there are free $\mathbb{Z}[t, t^{-1}]$-modules $F_0 = \langle u_1, \ldots, u_n \rangle$ and $F_1 = \langle v_1, \ldots, v_m \rangle$ that fit into an exact sequence of the form

$$F_0 \to F_1 \to H_1(C^\infty) \to 0.$$

Suppose in this presentation the image of u_j in F_1 is given by $\Sigma_{i=1}^m a_{i,j} v_i$. The $m \times n$ matrix $A = (a_{i,j})$ is called a *presentation matrix* for $H_1(C^\infty)$. If $m \leq n$ then the *reduced Alexander polynomial* $\Delta_L(t)$ of the link L is defined to be a generator of the smallest principle ideal that contains the ideal generated by all the $m \times m$ minors of A. Hence, if A is square then $\Delta_L(t) = \det A$. If A is an $(n-1) \times n$ matrix, as it will be for us in the next section, then the reduced Alexander polynomial of L is the g.c.d. of all the $(n-1)$ by $(n-1)$ minors of A. Notice that since $\Delta_L(t)$ is defined to be any generator of a certain principle ideal of $\mathbb{Z}[t, t^{-1}]$ it is well defined only up to multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$.

4.2 Finding a Presentation Matrix for $H_1(C^\infty)$

We use the Mayer-Vietoris long exact sequence to give a presentation of the $\mathbb{Z}[t, t^{-1}]$-module $H_1(C^\infty)$. The reader will readily notice that what is being done in this section is quite similar to what was done in the proof of Theorem 5.

Notice first that in our construction above we could have taken C to be the closure of $S - N$, where $S \cong D \times I$ is a cylinder containing the fattened link N. Making this change will not affect $H_1(C^\infty)$. Hence, we shall think of C^∞ as having been defined in this manner.

Looking at Figure 4.1 we see that the space $C = \overline{S - N}$ can be viewed as having three components. There are the caps A and B at either end, which are both homotopy equivalent to the space X defined in section 3.1; and there is the middle part T, which is the part in which all the braiding occurs. Let Y be the space defined in section 3.1 obtained from the 2-disk D by removing $2n$ disjoint open disks. The space T has a copy of Y at each end, call them Y_0 and Y_1. The Seifert surface we have drawn in Figure 4.1 intersects Y_0 in n disjoint arcs $A_1, \ldots, A_n$ where the arc A_i joins the ith and the $(2n - i + 1)^{\text{st}}$ punctures. In fact, any closed braid has such a Seifert surface. It therefore follows from our construction of C^∞ that the induced coverings $\tilde{Y}_0 \to Y_0$ and $\tilde{Y}_1 \to Y_1$ are the same as the covering $\tilde{Y} \to Y$ constructed in section 3.1.

Let $j^0, j^1 : Y \to C$ be homeomorphisms taking Y to Y_0 and Y to Y_1, respectively. Let $x_1, \ldots, x_n, y_1, \ldots, y_n$ be the loops in Y drawn in Figure 2.6. Let $\alpha : I \to T$ be a path from $i_0(d_0)$ to $i_1(d_0)$. We specify $j^0(d_0)$ to be the base point of C. We define $x_i^0 = j^0 x_i$, and $y_i^0 = j^0 y_i$, which are loops in Y_0 based at $j^0 d_0$. Analogously, we let $x_i^1 = \alpha(j^1 x_i)\alpha^{-1}$ and $y_i^1 = \alpha(j^1 y_i)\alpha^{-1}$.

Suppose the braid $\beta \in \mathrm{Aut}(F_n)$ acts on F_n by (2.6)-(2.7). From the discussion which follows Theorem 4 we have that x_i is homotopic to $w_i^1 x_{\mu_i}^0 (w_i^1)^{-1}$ inside T. It should be clear from Figure 2.5 that y_i^0 is homotopic to y_i^1 inside T.

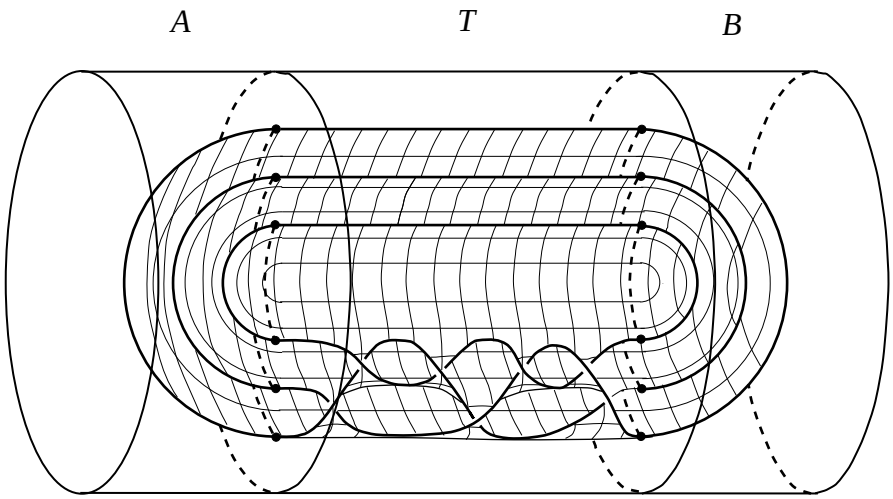


Figure 4.1: The complement in S^3 of a closed braid with its Seifert Surface

Let $U = T \cup B$. We have $C^\infty = \tilde{A} \cup \tilde{U}$ where $\tilde{A} \to A$ and $\tilde{U} \to U$ are the coverings induced by $C^\infty \to C$. Choose once and for all a lift $\tilde{d}_0$ of $j^0(d_0)$ to C^∞.

The retraction of A onto X induces a retraction of the induced covering $\tilde{A}$ onto $\tilde{X}$. Hence, the inclusion $\tilde{X} \hookrightarrow \tilde{A}$ induces an isomorphism on homology. So by Proposition 6 we have that $H_1(\tilde{A})$ is a free $\mathbb{Z}[t, t^{-1}]$-module generated by $v_1, \ldots, v_{n-1}$ where v_i is the lift of the loop $x_i^0(x_{i+1}^0)^{-1}$ relative to the base point $\tilde{d}_0$. Similarly, $H_1(\tilde{U})$ is a free $\mathbb{Z}[t, t^{-1}]$-module generated by $v_1', \ldots, v_{n-1}'$ where v_i' is a lift of the loop $x_i^1(x_{i+1}^1)^{-1}$ relative to $\tilde{d}_0$.

The intersection $A \cap U$ is homotopy equivalent to Y_0 by an obvious deformation retraction. In order to simplify notation let us regard $A \cap U$ as actually being equal to Y_0. In fact, the retraction $A \cap U \to Y_0$ lifts to a retraction on the induced coverings so that the inclusion $\tilde{Y}_0 \hookrightarrow \widetilde{A \cap U}$ induces an isomorphism on homology.

Applying Mayer-Vietoris gives us the following exact sequence:

$$H_1(\tilde{Y}_0) \xrightarrow{\zeta} H_1(\tilde{A}) \oplus H_1(\tilde{U}) \longrightarrow H_1(C^\infty) \xrightarrow{\delta} H_0(\tilde{Y}_0) \longrightarrow \cdots \tag{4.2}$$

The map $H_0(\tilde{Y}_0) \to H_0(\tilde{A}) \oplus H_0(\tilde{U})$ is an injection. Hence, we have $\delta = 0$. We have

already remarked that $H_1(\tilde{A})$ and $H_1(\tilde{U})$ are free over $\mathbb{Z}[t, t^{-1}]$. Similarly, Proposition 7 shows that $H_1(\tilde{Y}_0)$ is also free over $\mathbb{Z}[t, t^{-1}]$ generated by $\nu_1, \ldots, \nu_{n-1}$, $\omega_1, \ldots, \omega_{n-1}$, and ρ. Hence, the exact sequence in (4.2) gives a presentation of $H_1(C^\infty)$.

The map ζ is given by

$$\zeta : \begin{cases} \nu_i \mapsto (v_i, (\psi_{\mathrm{r}}\beta)v_i') \\ \omega_i \mapsto (v_i, v_i') \\ \rho \mapsto (0, v(\beta)) \end{cases} \tag{4.3}$$

so that we have the following presentation matrix for $H_1(C^\infty)$:

$$\left[\begin{array}{c|c|c} I_{n-1} & I_{n-1} & 0 \\ \hline \psi_{\mathrm{r}}\beta & I_{n-1} & v(\beta) \end{array}\right] \tag{4.4}$$

By applying a sequence of matrix moves to (4.4) (see refLic91 Theorem 6.1) we obtain the following presentation matrix for $H_1(C^\infty)$:

$$\left[\begin{array}{c|c} \psi_{\mathrm{r}}\beta - I & v(\beta) \end{array}\right] \tag{4.5}$$

4.3 Proof of Theorem 11

(4.5) gives us an $(n-1) \times n$ presentation matrix of the $\mathbb{Z}[t, t^{-1}]$-module $H_1(C^\infty)$. Let $D_k(t)$ be the kth minor of this matrix. That is, $D_k(t)$ is the determinant of the matrix obtained by deleting the kth column of $[\psi_{\mathrm{r}}\beta - I | v(\beta)]$. The g.c.d. of these polynomials is $\Delta_{\hat{\beta}}(t)$.

Let $v_0 = (a_0, \ldots, a_{n-1})$ where $a_j = (1 + t + \cdots + t^{j-1})/(1 + t + \cdots + t^{n-1})$. We think of v_0 as a column vector. We wish to show that for all braids β in B_n we have

$$(\psi_{\mathrm{r}}\beta - I)v_0 = v(\beta) \tag{4.6}$$

An easy calculation using (3.2) and (3.12) shows that (4.6) holds for $\sigma_1, \ldots, \sigma_{n-1}$.

Now, suppose β_1 and β_2 are two braids for which (4.6) holds. That is, $(\psi_{\mathrm{r}}\beta_1)v_0 = v\beta_1 + v_0$ and $(\psi_{\mathrm{r}}\beta_2)v_0 = v\beta_2 + v_0$. Then we have

$$(\psi_{\mathrm{r}}(\beta_1\beta_2) - I)v_0 = (\psi_{\mathrm{r}}\beta_1)(\psi_{\mathrm{r}}\beta_2)v_0 - v_0 \tag{4.7}$$

$$= (\psi_{\mathrm{r}}\beta_1)(v\beta_2 + v_0) - v_0 \tag{4.8}$$

$$= (\psi_{\mathrm{r}}\beta_1)(v\beta_2) + (\psi_{\mathrm{r}}\beta_1)v_0 - v_0 \tag{4.9}$$

$$= (\psi_{\mathrm{r}}\beta_1)(v\beta_2) + v\beta_1 = v(\beta_1\beta_2) \tag{4.10}$$

where the last equality follows from (3.11). Hence, (4.6) does indeed hold for all $\beta \in B_n$ so that we may write $[\psi_{\mathrm{r}}\beta - I|v\beta] = [\psi_{\mathrm{r}}\beta - I|(\psi_{\mathrm{r}}\beta - I)v_0]$.

Now, for $k < n$ the determinant of the matrix obtained by deleting the kth column of $[\psi_{\mathrm{r}}\beta - I|v\beta]$ is (up to a sign) equal to the determinant of the matrix obtained by replacing the kth column of $\psi_{\mathrm{r}}\beta - I$ with $v\beta$. But as $v\beta = (\psi_{\mathrm{r}}\beta - I)v_0$, this is equal to the determinant of $(\psi_{\mathrm{r}}\beta - I)T_k$ where T_k is the matrix which has v_0 in its kth column and is equal to the identity everywhere else. Hence, we have (up to a sign)

$$D_k(t) = \det((\psi_{\mathrm{r}}\beta - I)T_k) = \det(\psi_{\mathrm{r}}\beta - I)a_k. \tag{4.11}$$

Setting $k = 1$ we have

$$D_1(t) = \frac{\det(\psi_{\mathrm{r}}\beta - I)}{1 + t + \cdots + t^{n-1}}. \tag{4.12}$$

From these last two equations it follows that for all $k = 1, \ldots, n$ we have

$$D_k = (1 + t + \cdots + t^{k+1})D_1(t). \tag{4.13}$$

From this it follows that $D_1(t)$ is the g.c.d. of the polynomials $D_1(t), \ldots, D_n(t)$. That is, $D_1(t) = \Delta_{\hat{\beta}}(t)$. Hence, Theorem 11 follows immediately from equation (4.12).

CHAPTER 5

REPRESENTATIONS OF B_n FROM $U_q(\mathfrak{s}l_2)$

In this chapter we show how the Burau and Krammer representations of B_n can be obtained from the quantum algebra $U_q(\mathfrak{s}l_2)$. In particular, we first develop the basic theory of braided bialgebras and show that these algebras give representations of B_n in a natural way. Then in sections 5.3 and 5.4 we show that in the case of $U_q(\mathfrak{s}l_2)$ the Burau and Krammer representations are summands of one of these natural representations.

5.1 Braided Bialgebras

Let k be a field. An *algebra over* k is a k-vector space A with two k-linear maps $\mu : A \otimes_k A \to A$ and $\eta : k \to A$ such that

$$\mu(\mu \otimes 1_A) = \mu(1_A \otimes \mu) \qquad \text{(Associativity)}$$

$$\mu(\eta \otimes 1_A) = \mu(1_A \otimes \eta) = 1_A \qquad \text{(Identity)}$$

where in the second equation $k \otimes_k A$ and $A \otimes_k k$ are identified with A in the standard way.

The first equality shows the multiplication μ to associative while the second shows the element $\eta(1) \in A$ to be a left and right unit for μ. That is, A has a unitary ring structure defined by $aa' = \mu(a \otimes a')$ and $1 = \eta(1)$. Moreover, the map $\eta : k \to A$ is a homomorphism of rings.

The algebra A is *commutative* if $\mu = \mu\tau_{A,A}$ where $\tau_{A,A} : A \otimes_k A \to A \otimes_k A$ is the

linear isomorphism defined by $\tau_{A,A} : a \otimes a' \mapsto a' \otimes a$. A *morphism* of algebras is a k-linear map $f : A \to A'$ satisfying

$$\mu'(f \otimes f) = f\mu \qquad \text{and} \qquad \eta' = f\eta \tag{5.1}$$

Let A and A' be k-algebras. There is an k-algebra structure on the tensor product $A \otimes_k A'$. Multiplication is given by $(\mu \otimes \mu')(1 \otimes \tau_{A,A'} \otimes 1)$ and the unit is given by the composition of $\eta \otimes \eta'$ with the canonical isomorphism $k \cong k \otimes k$.

A *coalgebra over* k is a k-vector space A with two linear maps $\Delta : A \to A \otimes_k A$ and $\epsilon : A \to k$ such that

$$(\Delta \otimes 1_A)\Delta = (1_A \otimes \Delta)\Delta \tag{Coassociativity}$$

$$(\epsilon \otimes 1_A)\Delta = (1_A \otimes \epsilon)\Delta = 1_A \tag{Coidentity}$$

The map Δ is called the *comultiplication* of the coalgebra. The map ϵ is called the *counit*. A *morphism* of coalgebras is an k-linear map $f : A \to A'$ satisfying

$$\Delta' f = (f \otimes f)\Delta \qquad \text{and} \qquad \epsilon' f = \epsilon \tag{5.2}$$

Furthermore, the coalgebra A is said to be *cocommutative* if $\Delta = \tau_{A,A}\Delta$.

A coalgebra structure can be put on the tensor product of two coalgebras. Namely, we define a comultiplication on $A \otimes_k A'$ by $(1 \otimes \tau_{A,A'} \otimes 1)(\Delta \otimes \Delta')$. The counit is given as the composition of the canonical isomorphism $k \otimes k \cong k$ with $\epsilon \otimes \epsilon'$.

Note that the field k itself is both an algebra and a coalgebra over itself in a trivial way. This leads us to our next definition.

Definition 12. *A k-module A having both algebra and coalgebra structures is called a k-*bialgebra *if these structures are compatible with each other in the sense that the linear maps $\Delta : A \to A \otimes_k A$ and $\epsilon : A \to k$ are in fact morphisms of algebras. Or, what is equivalent, the linear maps $\mu : A \otimes_k A \to A$ and $\eta : k \to A$ are morphisms of coalgebras.*

A linear map $f : A \to A'$ from one bialgebra to another is a *morphism* of bialgebras if it is both a morphism of algebras and a morphism of coalgebras.

Let $(A, \mu, \eta, \Delta, \epsilon)$ be a bialgebra. Since any algebra A is a ring via μ we may consider the category A-$\mathcal{Mod}$ of all left A-modules. Any A-module is also a vector space over k so we can take the tensor product $U \otimes_k V$ over k of any two A-modules U and V. This product is an $A \otimes_k A$-module via $(a \otimes a')(u \otimes v) = au \otimes a'v$. The coproduct allows us to equip $U \otimes_k V$ with an A-module structure by $a(u \otimes v) = \Delta(a)(u \otimes v)$. Furthermore, the counit equips k with an A-module structure by $ax = \epsilon(a)x$. For three A-modules U, V, W we have the canonical k-linear isomorphisms

$$(U \otimes_k V) \otimes_k W \cong U \otimes_k (V \otimes_k W) \tag{5.3}$$

$$k \otimes_k V \cong V \cong V \otimes_k k \tag{5.4}$$

which are shown to be A-linear by the coassociativity and coidentity relations for δ and ϵ, respectively. Furthermore, if $f : V \to V'$ and $g : W \to W'$ are two A-linear homomorphisms, then the map $f \otimes_k g : V \otimes_k V' \to W \otimes_k W'$ is also A-linear. Hence, the functor $\otimes_k$ makes A-$\mathcal{Mod}$ into a monoidal category (for the formal definition of a monoidal category see [12]). From here on out all tensor products will be taken with respect to k so we shall omit the k subscript from the tensor product symbol.

Definition 13. *Let $(A, \mu, \eta, \Delta, \epsilon)$ be a k-bialgebra. A* commutativity constraint *c in the category A-$\mathcal{Mod}$ is a family of isomorphisms $c_{V,W} : V \otimes W \to W \otimes V$ defined for all pairs of A-modules V and W such that the following diagram commutes for all A-linear maps f, g:*

$$\begin{array}{ccc} V \otimes W & \xrightarrow{c_{V,W}} & W \otimes V \\ {\scriptstyle f \otimes g}\downarrow & & \downarrow{\scriptstyle g \otimes f} \\ V' \otimes W' & \xrightarrow{c_{V',W'}} & W' \otimes V' \end{array} \tag{5.5}$$

Definition 14. *Let $(A, \mu, \eta, \Delta, \epsilon)$ be an k-bialgebra. A* braiding *in the category A-$\mathcal{M}od$ is a commutativity constraint c which satisfies the following two relations for all A-modules U, V, W:*

$$c_{U\otimes V,W} = (c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W}) \tag{5.6}$$

$$c_{U,V\otimes W} = (1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W) \tag{5.7}$$

A bialgebra $(A, \mu, \eta, \Delta, \epsilon)$ whose category of modules has a braiding will be called a *braided bialgebra*. Any cocommutative bialgebra is braided, the braiding being given by the flip isomorphism τ. We will see an example of a non-cocommutative braided bialgebra in the next section. For now we have the following theorem which characterizes those bialgebras which admit braidings.

Theorem 15. *A k-bialgebra $(A, \mu, \eta, \Delta, \epsilon)$ is braided if and only if there is an invertible element R of $A \otimes A$ such that*

$$\tau_{A,A}\Delta(x) = R\Delta(x)R^{-1} \qquad \textit{for all } x \in A \tag{5.8}$$

$$(\Delta \otimes 1_A)(R) = R_{13}R_{23} \tag{5.9}$$

$$(1_A \otimes \Delta)(R) = R_{13}R_{12} \tag{5.10}$$

where $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, and $R_{13} = (\tau_{A,A} \otimes 1_A)(1 \otimes R)$.

Proof. We will give the proof of sufficiency. For necessity refer to [12], page 19.

Let $R \in A \otimes A$ be as in the statement of the theorem. Let V and W be left A-modules. Define $c^R_{V,W} : V \otimes W \to W \otimes V$ by

$$c^R_{V,W}(v \otimes w) = \tau_{V,W}(R(v \otimes w)) \tag{5.11}$$

for all $v \in V$ and $w \in W$. We claim that the family $\{c^R_{V,W}\}_{V,W}$ is a braiding in the category A-$\mathcal{M}od$.

Let x be in A, then using (5.8) we have

$$\begin{aligned} c^R_{V,W}(x(v \otimes w)) &= \tau_{V,W}(R\Delta(x)(v \otimes w)) \\ &= \tau_{V,W}(\tau_{A,A}\Delta(x)R(v \otimes w)) \\ &= \Delta(x)\tau_{V,W}(R(v \otimes w)) \\ &= x(c^R_{V,W}(v \otimes w)) \end{aligned}$$

so that $c^R_{V,W}$ is A-linear.

It is clear that $c^R_{V,W}$ satisfies (5.5) and that it has inverse given by $(c^R_{V,W})^{-1} = R^{-1}\tau_{W,V}$. Hence, we need only verify conditions (5.6) and (5.7). Using (5.9) we have:

$$\begin{aligned} (c^R_{U,W} \otimes 1_V)(1_U \otimes c^R_{V,W}) &= (\tau_{U,W}R \otimes 1_V)(1_U \otimes \tau_{V,W}R) \\ &= (\tau_{U,W} \otimes 1_V)(R \otimes 1)(1_U \otimes \tau_{V,W})(1 \otimes R) \\ &= (\tau_{U,W} \otimes 1_V)(1_U \otimes \tau_{V,W})[(\tau_{A,A} \otimes 1_A)(1 \otimes R)](1 \otimes R) \\ &= (\tau_{U\otimes V,W})(\Delta \otimes 1_A)(R) \\ &= c^R_{U\otimes V,W} \end{aligned}$$

Using (5.10) we have:

$$\begin{aligned} (1_V \otimes c^R_{U,W})(c^R_{U,V} \otimes 1_W) &= (1_V \otimes \tau_{U,W}R)(\tau_{U,V}R \otimes 1_W) \\ &= (1_V \otimes \tau_{U,W})(1 \otimes R)(\tau_{U,V} \otimes 1_W)(R \otimes 1) \\ &= (1_V \otimes \tau_{U,W})(\tau_{U,V} \otimes 1_W)[(\tau_{A,A} \otimes 1_A)(1 \otimes R)](R \otimes 1) \\ &= (\tau_{U,V\otimes W})(1_A \otimes \Delta)(R) \\ &= c^R_{U,V\otimes W} \end{aligned}$$

□

An invertible element $R \in A \otimes A$ is called a *universal R-matrix*.

The notion of a braiding will allow us us to construct representations of B_n in a systematic way. The following theorem provides the key result we need in order to achieve this.

Theorem 16. *Let $(A, \mu, \eta, \Delta, \epsilon)$ be an R-bialgebra and let c be a braiding in the category A-$\mathcal{M}od$. Then for all A-modules U, V, and W we have:*

$$(c_{V,W} \otimes 1_U)(1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W) = (1_W \otimes c_{U,V})(c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W}) \tag{5.12}$$

Proof. We have

$$\begin{aligned}(c_{V,W} \otimes 1_U)(1_V \otimes c_{U,W})(c_{U,V} \otimes 1_W) &= (c_{V,W} \otimes 1_U)c_{U,V\otimes W} \\ &= c_{U,W\otimes V}(1_U \otimes c_{U,W\otimes V}) \\ &= (1_W \otimes c_{U,V})(c_{U,W} \otimes 1_V)(1_U \otimes c_{V,W})\end{aligned}$$

The first and last equalities follow from (5.7) while the middle equality follows from (5.5) with $f = 1_U$ and $g = c_{V,W}$. □

Now if A is any braided bialgebra with braiding c (for instance, $c = c^R$ as in (5.11) where R is a universal R-matrix for A) and V is any A-module, then we may define $c_i = 1 \otimes \cdots \otimes c_{V,V} \otimes \cdots \otimes 1$, an automorphism of the n-fold tensor product $V^{\otimes n}$, where the $c_{V,V}$ term occupies the ith and $(i+1)^{\text{st}}$ places. Theorem 16 then implies that $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$. That is, the $n-1$ elements $c_1, \ldots, c_{n-1} \in \mathrm{Aut}(V^{\otimes n})$ satisfy the braid relations (2.1) - (2.2). Hence, for each A-module V we obtain a representation $\rho_V : B_n \to \mathrm{GL}(V^{\otimes n})$ given by $\sigma_i \mapsto c_i$.

In the next section we study the representations of B_n given by the universal R-matrix of the braided bialgebra $U_q(\mathfrak{sl}_2)$. In particular, we will see that both the Burau and Krammer representations may be given in this way.

5.2 $U_q(\mathfrak{sl}_2)$

Let k be a field and let $q \in k$ with $q \neq 0$, and $q^2 \neq 1$. The quantum algebra $U_q(\mathfrak{sl}_2)$ is defined to be the associative algebra with 1 generated by E, F, K, and K^{-1} with relations given by

$$KK^{-1} = K^{-1}K = 1 \tag{5.13}$$

$$KEK^{-1} = q^2 E \tag{5.14}$$

$$KFK^{-1} = q^{-2} F \tag{5.15}$$

$$[E, F] = \frac{K - K^{-1}}{q - q^{-1}} \tag{5.16}$$

The idea is that $U_q(\mathfrak{s}l_2)$ is a quantum deformation of the Lie algebra $\mathfrak{s}l_2$ of complex 2×2 matrices of trace 0. This algebra $\mathfrak{s}l_2$ is generated by the elements E, F, and H with relations $[H, E] = 2E$, $[H, F] = -2F$, and $[E, F] = H$. To deform $\mathfrak{s}l_2$ into $U_q(\mathfrak{s}l_2)$ we allow infinite formal sums. Setting $K = q^H$ as a formal power series the relations for $\mathfrak{s}l_2$ will then give, modulo subtleties, the relations (5.13)–(5.16). For a more thorough discussion see [11], Chapters 7 and 17.

Define a comultiplication $\Delta : U_q(\mathfrak{s}l_2) \to U_q(\mathfrak{s}l_2) \otimes U_q(\mathfrak{s}l_2)$ by

$$\Delta(E) = E \otimes 1 + K \otimes E \tag{5.17}$$

$$\Delta(F) = F \otimes K^{-1} + 1 \otimes F \tag{5.18}$$

$$\Delta(K) = K \otimes K \tag{5.19}$$

Also, define a counit $\epsilon : U_q(\mathfrak{s}l_2) \to k$ by $\epsilon : K, K^{-1} \mapsto 1$, and $\epsilon : E, F \mapsto 0$. It is easy to check that these co-operations give $U_q(\mathfrak{s}l_2)$ a bialgebra structure. In fact, $U_q(\mathfrak{s}l_2)$ is a braided bialgebra.

Let $[n]_q$ denote the element $\frac{q^n - q^{-n}}{q - q^{-1}}$ in k and let $[n]_q! = [n]_q[n-1]_q \dots [1]_q$. We have the following universal R-matrix for $U_q(\mathfrak{s}l_2)$ (see for instance [7] or [10]):

$$R \;=\; \Big(\sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} \frac{(1-q^2)^n}{[n]_q!} E^n \otimes F^n\Big) q^{-\frac{1}{2}(H \otimes H)} \tag{5.20}$$

Hence, by the results of the previous section, for any $U_q(\mathfrak{s}l_2)$-module V we obtain a representation $\rho : B_n \to \mathrm{GL}(V^{\otimes n})$ defined by $\rho : \sigma_i \mapsto c_i$ where $c_i = 1 \otimes \cdots \otimes c^R_{V,V} \otimes \cdots \otimes 1$ and $c^R_{V,V}$ is given by (5.11).

Let $\lambda \neq 0$ be in k and let $V_\lambda = \langle v_0, v_1, \dots \rangle$ be the k-vector space generated by the

distinct elements $\{v_0, v_1, \dots\}$. We make V_λ into a $U_q(\mathfrak{sl}_2)$-module by letting $U_q(\mathfrak{sl}_2)$ act on V_λ as follows:

$$Kv_l = q^{\lambda-2l}v_l \tag{5.21}$$

$$Fv_l = v_{l+1} \tag{5.22}$$

$$Ev_l = [l]_q[\lambda+1-l]_q v_{l-1} \tag{5.23}$$

Let c denote the automorphism $c^R_{V,V}$ where R is given by (5.20) and $V = V_\lambda$. A direct calculation gives

$$c(v_0 \otimes v_0) = q^{-\frac{1}{2}\lambda^2} v_0 \otimes v_0 \tag{5.24}$$

$$c(v_0 \otimes v_1) = q^{-\frac{1}{2}\lambda(\lambda-2)} v_1 \otimes v_0 \tag{5.25}$$

$$c(v_1 \otimes v_0) = q^{-\frac{1}{2}\lambda(\lambda-2)}\Big[v_0 \otimes v_1 + (q^{-\lambda} - q^{\lambda})v_1 \otimes v_0\Big] \tag{5.26}$$

$$c(v_1 \otimes v_1) = q^{-\frac{1}{2}(\lambda-2)^2}\Big[v_1 \otimes v_1 + (q^{-\lambda} - q^{\lambda})v_2 \otimes v_0\Big] \tag{5.27}$$

$$\begin{aligned} c(v_2 \otimes v_0) = q^{-\frac{1}{2}\lambda(\lambda-4)}\Big[&v_0 \otimes v_2 + (q+q^{-1})(q^{-\lambda+1} - q^{\lambda-1})v_1 \otimes v_1 \\ &+ q^{-1}(q^{\lambda-1} - q^{-\lambda+1})(q^{\lambda} - q^{-\lambda})v_2 \otimes v_0\Big] \end{aligned} \tag{5.28}$$

$$c(v_0 \otimes v_2) = q^{-\frac{1}{2}\lambda(\lambda-4)}(v_2 \otimes v_0) \tag{5.29}$$

5.3 The Burau Representation from $U_q(\mathfrak{sl}_2)$

Let $(V^{\otimes n})_1$ be the n-dimensional subspace of $V^{\otimes n}$ generated by $\{\hat{u}_1, \dots, \hat{u}_n\}$ where $\hat{u}_i = v_0 \otimes \cdots \otimes v_1 \otimes \cdots \otimes v_0$, the vector v_1 occurring in the ith position. Equations (5.24)–(5.26) imply that

$$(\rho\sigma_i)\hat{u}_j = q^{-\frac{1}{2}\lambda^2}\hat{u}_j \qquad \text{for } j \neq i, i+1 \tag{5.30}$$

$$(\rho\sigma_i)\hat{u}_{i+1} = q^{-\frac{1}{2}\lambda(\lambda-2)}\hat{u}_i \tag{5.31}$$

$$(\rho\sigma_i)\hat{u}_i = q^{-\frac{1}{2}\lambda(\lambda-2)}\Big[\hat{u}_{i+1} + (q^{-\lambda} - q^{\lambda})\hat{u}_i\Big] \tag{5.32}$$

so that the representation $\rho : B_n \to \mathrm{GL}(V^{\otimes n})$ restricts to a representation $\rho_1 : B_n \to \mathrm{GL}((V^{\otimes n})_1)$.

We normalize the representation $\rho_1 : B_n \to \mathrm{GL}((V^{\otimes n})_1)$ by setting $\rho_1 = q^{\frac{1}{2}\lambda^2}\rho_1$. Then equations (5.30) - (5.32) become

$$(\rho_1\sigma_i)\hat{u}_j = \hat{u}_j \qquad \text{for } j \neq i, i+1 \tag{5.33}$$

$$(\rho_1\sigma_i)\hat{u}_{i+1} = q^\lambda \hat{u}_i \tag{5.34}$$

$$(\rho_1\sigma_i)\hat{u}_i = q^\lambda \hat{u}_{i+1} + (1-q^{2\lambda})\hat{u}_i. \tag{5.35}$$

Let $W_1 \subseteq (V^{\otimes n})_1$ be the k-vector subspace generated by $\{u_i\}_{1\leq i\leq n-1}$ where

$$u_i = q^\lambda \hat{u}_i - \hat{u}_{i+1}. \tag{5.36}$$

Then equations (5.33) - (5.35) then give the following:

$$(\rho_1\sigma_i)u_j = u_j \qquad \text{for } j \neq i-1, i, i+1 \tag{5.37}$$

$$(\rho_1\sigma_i)u_{i+1} = q^\lambda q^\lambda \hat{u}_i - \hat{u}_{i+2} = q^{2\lambda}\hat{u}_i - q\hat{u}_{i+1} + q\hat{u}_{i+1} - \hat{u}_{i+2} = q^\lambda u_i + u_{i+1} \tag{5.38}$$

$$(\rho_1\sigma_i)u_i = q^\lambda\left(q^\lambda\hat{u}_{i+1} + (1-q^{2\lambda})\hat{u}_i\right) - q^\lambda\hat{u}_i = q^{2\lambda}\hat{u}_{i+1} - q^{3\lambda}\hat{u}_i = -q^{2\lambda}u_i \tag{5.39}$$

$$\begin{aligned}(\rho_1\sigma_i)u_{i-1} &= q^\lambda\hat{u}_{i-1} - \left(q^\lambda\hat{u}_{i+1} + (1-q^{2\lambda})\hat{u}_i\right)\\ &= q^\lambda\hat{u}_{i-1} - \hat{u}_i + q^\lambda(q^\lambda\hat{u}_i - \hat{u}_{i+1}) = u_{i-1} + q^\lambda u_i\end{aligned} \tag{5.40}$$

Hence, $\rho_1 : B_n \to \mathrm{GL}((V^{\otimes n})_1)$ restricts to a representation $\rho_1 : B_n \to \mathrm{GL}(W_1)$. If we now rescale the basis $\{u_i\}_{1\leq i\leq n-1}$ of W_1 by multiplying each u_i by $q^{-\lambda i}$, then in this new basis equations (5.37)–(5.40) reduce to

$$(\rho_1\sigma_i)u_j = u_j \qquad \text{for } j \neq i-1, i, i+1 \tag{5.41}$$

$$(\rho_1\sigma_i)u_{i+1} = u_i + u_{i+1} \tag{5.42}$$

$$(\rho_1\sigma_i)u_i = -q^{2\lambda}u_i \tag{5.43}$$

$$(\rho_1\sigma_i)u_{i-1} = u_{i-1} + q^{2\lambda})u_i \tag{5.44}$$

Setting $t = q^{2\lambda}$ we have the reduced Burau representation as given by (3.1).

5.4 The Krammer Representation from $U_q(\mathfrak{sl}_2)$

For $1 \leq i < j \leq n$ let $\hat{w}_{i,j} = v_0 \otimes \cdots \otimes v_1 \otimes \cdots \otimes v_1 \otimes \cdots \otimes v_0$, where the two v_1 terms occur in the ith and j^{th} positions, respectively. For $0 \leq i \leq n$ let $\hat{w}_i =$

$v_0 \otimes \cdots \otimes v_2 \otimes \cdots \otimes v_0$, the v_2 term occurring in the ith position.. Let $(V^{\otimes n})_2$ be the $(\binom{n}{2} + n)$-dimensional subspace of $V^{\otimes n}$ generated by $\{\hat{w}_1, \ldots, \hat{w}_n, \hat{w}_{1,2}, \ldots, \hat{w}_{n-1,n}\}$. Let $\{j,k\} \cap \{i, i+1\} = \emptyset$, then equations (5.24)–(5.29) imply that the representation $\rho : B_n \to \mathrm{GL}(V^{\otimes n})$ restricts to a representation $\rho_2 : B_n \to \mathrm{GL}((V^{\otimes n})_2)$ given by

$$(\rho_2\sigma_i)\hat{w}_{j,k} = q^{-\frac{1}{2}\lambda^2}\hat{w}_{j,k} \tag{5.45}$$

$$(\rho_2\sigma_i)\hat{w}_{i+1,j} = q^{-\frac{1}{2}\lambda(\lambda-2)}\hat{w}_{i,j} \tag{5.46}$$

$$(\rho_2\sigma_i)\hat{w}_{i,j} = q^{-\frac{1}{2}\lambda(\lambda-2)}\Big[\hat{w}_{i+1,j} + (q^{-\lambda} - q^{\lambda})\hat{w}_{i,j}\Big] \tag{5.47}$$

$$(\rho_2\sigma_i)\hat{w}_{i,i+1} = q^{-\frac{1}{2}(\lambda-2)^2}\Big[\hat{w}_{i,i+1} + (q^{-\lambda} - q^{\lambda})\hat{w}_{i}\Big] \tag{5.48}$$

$$(\rho_2\sigma_i)\hat{w}_{j} = q^{-\frac{1}{2}\lambda^2}\hat{w}_{j} \tag{5.49}$$

$$\begin{aligned}(\rho_2\sigma_i)\hat{w}_{i} = q^{-\frac{1}{2}\lambda(\lambda-4)}\Big[&\hat{w}_{i+1} + (q+q^{-1})(q^{-\lambda+1} - q^{\lambda-1})\hat{w}_{i,i+1} \\ &+ q^{-1}(q^{\lambda-1} - q^{-\lambda+1})(q^{\lambda} - q^{-\lambda})\hat{w}_{i}\Big]\end{aligned} \tag{5.50}$$

$$(\rho_2\sigma_i)\hat{w}_{i+1} = q^{-\frac{1}{2}\lambda(\lambda-4)}\hat{w}_{i} \tag{5.51}$$

For each pair of integers $\{i,j\}$ with $0 \le i < j \le n$ we define the following elements of k:

$$\alpha_{i,j} = q^{\lambda(i-j)}\frac{[\lambda]}{[2][\lambda-1]} \tag{5.52}$$

$$\beta_{i,j} = q^{\lambda(j-i)-2}\frac{[\lambda]}{[2][\lambda-1]} \tag{5.53}$$

Let $W_2 \subseteq (V^{\otimes n})_2$ be the $\binom{n}{2}$-dimensional subspace generated by $\{w_{i,j}\}_{i<j}$ for $0 \le i < j \le n$ where

$$w_{i,j} = \hat{w}_{i,j} - \alpha_{i,j}\hat{w}_j - \beta_{i,j}\hat{w}_i. \tag{5.54}$$

We would like to show that $\rho_2 : B_n \to \mathrm{GL}((V^{\otimes n})_2)$ restricts to a representation $\rho_2 : B_n \to \mathrm{GL}(W_2)$, and that this restricted representation is equivalent to the Krammer representation.

We normalize the representation $\rho_2 : B_n \to \mathrm{GL}((V^{\otimes n})_2)$ by setting $\rho_2 = q^{\frac{1}{2}\lambda^2}\rho_2$. Also, in order to simplify our equations let us set $\Omega_i = q^{\frac{1}{2}\lambda^2}(\rho_2\sigma_i)\hat{w}_i$ as given in (5.50).

Specifying $\{j,k\} \cap \{i,i+1\} = \emptyset$, we may then calculate the action of $\rho_2\sigma_i$ on the elements $w_{i,j} \in W_2$ directly from equations (5.54) and (5.45)–(5.51):

$$(\rho_2\sigma_i)w_{j,k} = w_{j,k} \tag{5.55}$$

$$(\rho_2\sigma_i)w_{i+1,j} = q^{\lambda}\hat{w}_{i,j} - \alpha_{i+1,j}\hat{w}_j - \beta_{i+1,j}q^{2\lambda}\hat{w}_i \tag{5.56}$$

$$(\rho_2\sigma_i)w_{j,i+1} = q^{\lambda}\hat{w}_{j,i} - \alpha_{j,i+1}q^{2\lambda}\hat{w}_i - \beta_{j,i+1}\hat{w}_j \tag{5.57}$$

$$(\rho_2\sigma_i)w_{i,j} = q^{\lambda}\Big[\hat{w}_{i+1,j} + (q^{-\lambda} - q^{\lambda})\hat{w}_{i,j}\Big] - \alpha_{i,j}\hat{w}_j - \beta_{i,j}\Omega_i \tag{5.58}$$

$$(\rho_2\sigma_i)w_{i,i+1} = q^{2\lambda-2}\Big[\hat{w}_{i,i+1} + (q^{-\lambda} - q^{\lambda})\hat{w}_i\Big] - \alpha_{i,i+1}q^{2\lambda}\hat{w}_i - \beta_{i,i+1}\Omega_i \tag{5.59}$$

$$(\rho_2\sigma_i)w_{j,i} = q^{\lambda}\Big[\hat{w}_{j,i+1} + (q^{-\lambda} - q^{\lambda})\hat{w}_{j,i}\Big] - \alpha_{j,i}\Omega_i - \beta_{j,i}\hat{w}_j \tag{5.60}$$

Proposition 17. *Equations* (5.55)–(5.60) *reduce to the following:*

$$(\rho_2\sigma_i)w_{j,k} = w_{j,k} \tag{5.61}$$

$$(\rho_2\sigma_i)w_{i+1,j} = q^{\lambda}w_{i,j} \tag{5.62}$$

$$(\rho_2\sigma_i)w_{j,i+1} = q^{\lambda}w_{j,i} \tag{5.63}$$

$$(\rho_2\sigma_i)w_{i,j} = q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda} - q^{-\lambda})w_{i,i+1} + (1 - q^{2\lambda})w_{i,j} + q^{\lambda}w_{i+1,j} \tag{5.64}$$

$$(\rho_2\sigma_i)w_{i,i+1} = q^{4\lambda-2}w_{i,i+1} \tag{5.65}$$

$$(\rho_2\sigma_i)w_{j,i} = (1 - q^{2\lambda})w_{j,i} + q^{\lambda}w_{j,i+1} + q^{\lambda(j-i)}q^{2\lambda}(q^{\lambda} - q^{-\lambda})w_{i,i+1} \tag{5.66}$$

Hence, the representation $\rho_2 : B_n \to GL((V^{\otimes n})_2)$ restricts to a representation $\rho_2 : B_n \to GL(W_2)$. Moreover, this representation is equivalent to the Krammer representation.

Proof. The proof of the first part of this proposition just involves simple algebraic manipulation of equations (5.55)–(5.60) using definitions (5.52)–(5.54) and the fact

that $\Omega_i = q^{\frac{1}{2}\lambda^2}(\rho_2\sigma_i)\hat{w}_i$. For instance, to get equation (5.64) we calculate the following:

$$
\begin{aligned}
(\rho_2\sigma_i)w_{i,j} &= q^{\lambda}\left[\hat{w}_{i+1,j} + (q^{-\lambda} - q^{\lambda})\hat{w}_{i,j}\right] - \alpha_{i,j}\hat{w}_j - \beta_{i,j}\Omega_i \\
&= q^{\lambda}(w_{i+1,j} + \alpha_{i+1,j}\hat{w}_j + \beta_{i+1,j}\hat{w}_{i+1}) \\
&\quad + (1 - q^{2\lambda})(w_{i,j} + \alpha_{i,j}\hat{w}_j + \beta_{i,j}\hat{w}_i) - \alpha_{i,j}\hat{w}_j \\
&\quad - \beta_{i,j}q^{2\lambda}(q + q^{-1})(q^{-\lambda+1} - q^{\lambda-1})(w_{i,i+1} + \alpha_{i,i+1}\hat{w}_{i+1} + \beta_{i,i+1}\hat{w}_i) \\
&\quad - \beta_{i,j}q^{2\lambda}\hat{w}_{i+1} - \beta_{i,j}q^{2\lambda}q^{-1}(q^{\lambda-1} - q^{-\lambda+1})(q^{\lambda} - q^{-\lambda})\hat{w}_i \\
&= q^{\lambda}w_{i+1,j} + (1 - q^{2\lambda})w_{i,j} + q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda} - q^{-\lambda})w_{i,i+1} \\
&\quad + \left[q^{\lambda}\alpha_{i+1,j} + (1 - q^{2\lambda})\alpha_{i,j} - \alpha_{i,j}\right]\hat{w}_j \\
&\quad + \left[q^{\lambda}\beta_{i+1,j} + q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda} - q^{-\lambda})\alpha_{i,i+1} - \beta_{i,j}q^{2\lambda}\right]\hat{w}_{i+1} \\
&\quad + \Big[(1 - q^{2\lambda})\beta_{i,j} + q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda} - q^{-\lambda})\beta_{i,i+1} \\
&\qquad\qquad - \beta_{i,j}q^{2\lambda}q^{-1}(q^{\lambda-1} - q^{-\lambda+1})(q^{\lambda} - q^{-\lambda})\Big]\hat{w}_i
\end{aligned}
$$

And the coefficients of $\hat{w}_j$, $\hat{w}_{i+1}$, and $\hat{w}_i$ are readily shown to be zero:

$$
q^{\lambda}\alpha_{i+1,j} + (1 - q^{2\lambda})\alpha_{i,j} - \alpha_{i,j} = q^{\lambda}\alpha_{i,j} - q^{\lambda}\alpha_{i,j} = 0 \tag{5.67}
$$

$$
\begin{aligned}
q^{\lambda}\beta_{i+1,j} &+ q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda} - q^{-\lambda})\alpha_{i,i+1} - \beta_{i,j}q^{2\lambda} && (5.68) \\
&= q^{\lambda}q^{-\lambda}\beta_{i,j} - q^{2\lambda}(q^{-1} - q)[2][\lambda - 1]\alpha_{i,i+1}\beta_{i,j} - \beta_{i,j}q^{2\lambda} && (5.69) \\
&= \beta_{i,j}\left[1 - q^{2\lambda}q^{-\lambda}[\lambda](q^{-1} - q) - q^{2\lambda}\right] && (5.70) \\
&= \beta_{i,j}\left[1 + q^{\lambda}(q^{\lambda} - q^{-\lambda}) - q^{2\lambda}\right] = 0 && (5.71)
\end{aligned}
$$

$$
\begin{aligned}
(1-q^{2\lambda})\beta_{i,j} + q^{\lambda(j-i)}q^{2\lambda-2}(q^{\lambda}-q^{-\lambda})\beta_{i,i+1} & \\
- \beta_{i,j}q^{2\lambda}q^{-1}(q^{\lambda-1}-q^{-\lambda+1})(q^{\lambda}-q^{-\lambda}) & \qquad (5.72)\\
= (1-q^{2\lambda})\beta_{i,j} - q^{2\lambda}(q^{-1}-q)[2][\lambda-1]\beta_{i,i+1}\beta_{i,j} & \\
- \beta_{i,j}q^{2\lambda}q^{-1}(q^{\lambda-1}-q^{-\lambda+1})(q^{\lambda}-q^{-\lambda}) & \qquad (5.73)\\
= \beta_{i,j}\Big[1-q^{2\lambda}-q^{2\lambda}(q^{-1}-q)[\lambda]q^{\lambda-2}-(q^{3\lambda-2}-q^{\lambda})(q^{\lambda}-q^{-\lambda})\Big] & \qquad (5.74)\\
= \beta_{i,j}\Big[1-q^{2\lambda}+q^{3\lambda-2}(q^{\lambda}-q^{-\lambda})+(q^{\lambda}-q^{3\lambda-2})(q^{\lambda}-q^{-\lambda})\Big] & \qquad (5.75)\\
= \beta_{i,j}\Big[1-q^{2\lambda}+(q^{\lambda}-q^{-\lambda})(q^{3\lambda-2}+q^{\lambda}-q^{3\lambda-2})\Big] = 0 & \qquad (5.76)
\end{aligned}
$$

The remaining equations follow in similar fashion.

If we now rescale the basis $\{w_{i,j}\}_{i<j}$ of W_2 by multiplying each $w_{i,j}$ by $q^{-\lambda(i+j)}$, then in this new basis equations (5.61)–(5.66) reduce to

$$
\begin{aligned}
(\rho_2\sigma_i)w_{j,k} &= w_{j,k} && (5.77)\\
(\rho_2\sigma_i)w_{i+1,j} &= w_{i,j} && (5.78)\\
(\rho_2\sigma_i)w_{j,i+1} &= w_{j,i} && (5.79)\\
(\rho_2\sigma_i)w_{i,j} &= q^{-2}q^{2\lambda}(q^{2\lambda}-1)w_{i,i+1} + (1-q^{2\lambda})w_{i,j} + q^{2\lambda}w_{i+1,j} && (5.80)\\
(\rho_2\sigma_i)w_{i,i+1} &= q^{-2}q^{4\lambda}w_{i,i+1} && (5.81)\\
(\rho_2\sigma_i)w_{j,i} &= (1-q^{2\lambda})w_{j,i} + q^{2\lambda}w_{j,i+1} + q^{2\lambda}(q^{2\lambda}-1)w_{i,i+1} && (5.82)
\end{aligned}
$$

Setting "$q = q^{2\lambda}$" and $t = -q^{-2}$ we obtain the Krammer representation (see Theorem 10). □

5.5 The Verma Module

Certain parts of the above discussion deserve elaboration. Specifically, how we came to define the vector spaces V_λ, W_1, and W_2. As it stands now these definitions are rather obscure, their only justification being that they eventually work out to give the representations we want. In what follows we a somewhat better explanation of how these definitions arise.

Let U_+ be the subalgebra of $U_q(\mathfrak{sl}_2)$ generated by the elements E, K, and K^{-1}. Let $\langle v_+ \rangle$ be the one-dimensional k-module with basis $\{v_+\}$. Give $\langle v_+ \rangle$ the structure of a U_+-module by letting

$$Ev_+ = 0 \qquad \text{and} \qquad Kv_+ = q^\lambda v_+ \tag{5.83}$$

The *Verma module* V is the induced module $V = U_q(\mathfrak{sl}_2) \otimes_{U_+} \langle v_+ \rangle$. The relations for $U_q(\mathfrak{sl}_2)$ given in eqrefqrel1–eqrefqrel4 allow us to calculate the action of $U_q(\mathfrak{sl}_2)$ on V. Specifically, for any $lgeq0$ we have

$$KF^l \otimes v_+ = q^{-2l} F^l K \otimes v_+ = q^{\lambda - 2l} F^l \otimes v_+ \tag{5.84}$$

so that

$$EF \otimes v_+ = (FE + \frac{K - K^{-1}}{q - q^{-1}}) \otimes v_+ = \frac{K \otimes v_+ - K^{-1} \otimes v_+}{q - q^{-1}} = [\lambda]_q (1 \otimes v_+). \tag{5.85}$$

By induction on l we obtain

$$EF^l \otimes v_+ = (FEF^{l-1} + \frac{K - K^{-1}}{q - q^{-1}} F^{l-1}) \otimes v_+ = FEF^{l-1} \otimes v_+ + \frac{K - K^{-1}}{q - q^{-1}} F^{l-1} \otimes v_+ \tag{5.86}$$

$$= F\left([l-1]_q [\lambda + 1 - (l-1)]_q F^{l-2} \otimes v_+\right) + [\lambda - 2(l-1)]_q F^{l-1} \otimes v_+ \tag{5.87}$$

$$= \left([l-1]_q [\lambda + 1 - (l-1)]_q + [\lambda - 2(i-1)]_q\right) F^{l-1} \otimes v_+ \tag{5.88}$$

$$= [l]_q [\lambda + 1 - l]_q F^{l-1} \otimes v_+ \tag{5.89}$$

Hence, setting $v_l = F^l \otimes v_+$ we see that the Verma module V_λ is precisely the module which we gave earlier in section 5.2.

Let us write $V = V_\lambda$. We may put a $U_q(\mathfrak{sl}_2)$-module structure on the n-fold tensor product $V^{\otimes n}$ by

$$xv = \Delta^{(n)}(x) v \tag{5.90}$$

where $x \in U_q(\mathfrak{sl}_2)$, $v \in V^{\otimes n}$, and $\Delta^{(n)}$ is defined by $\Delta^{(2)} = \Delta$, $\Delta^{(3)} = (\Delta \otimes 1)\Delta$, etc. which, by cocommutativity, is well defined no matter in which order the Δ's are applied.

Let R be the universal R-matrix for $U_q(\mathfrak{sl}_2)$ given by (5.20) and let $c^R_{V,V}$ be the automorphism of $V \otimes V$ defined by (5.11). Let $c_i = 1 \otimes \cdots \otimes c^R_{V,V} \otimes \cdots \otimes 1$ as in section 5.2. Theorem 15 shows that $c^R_{V,V}$ is $U_q(\mathfrak{sl}_2)$-linear. Hence, each c_i is $U_q(\mathfrak{sl}_2)$-linear as a map on $V^{\otimes n}$, which is a $U_q(\mathfrak{sl}_2)$-module by (5.90).

For each $x \in U_q(\mathfrak{sl}_2)$ the element $\Delta^{(n)}(x)$ acts on $V^{\otimes n}$ as a k-linear transformation. Consider the elements

$$\Delta^{(n)}(E) = \Sigma_{k=1}^{n}(K \otimes \cdots \otimes K \otimes E \otimes 1 \otimes \cdots \otimes 1) \tag{5.91}$$

$$\Delta^{(n)}(K - q^{n\lambda-2l}) = K \otimes \cdots \otimes K - q^{n\lambda-2l} 1 \otimes \cdots \otimes 1. \tag{5.92}$$

For each $l = 0, 1, \ldots$ let $W_l \subseteq V^{\otimes n}$ be the k-submodule given by

$$W_l = \ker\left(\Delta^{(n)}(E)\right) \cap \ker\left(\Delta^{(n)}(K - q^{n\lambda-2l})\right). \tag{5.93}$$

The vectors in $\ker\left(\Delta^{(n)}(K - q^{n\lambda-2l})\right)$ are said to have *weight* equal to l. This submodule is generated by all vectors $v_{l_1} \otimes \cdots \otimes v_{l_n}$ with $\Sigma_{i=1}^{n} l_i = l$. The vectors in W_l are called *highest weight vectors* and W_l itself is called a *highest weight space*. The linearity of c_i implies that W_l is invariant under c_i. Hence, the representation $\rho : B_n \to \mathrm{Aut}(V^{\otimes n})$ defined by $\sigma_i \mapsto c_i$ will restrict for all $l = 0, 1, \ldots$ to a representation $\rho_l : B_n \to \mathrm{Aut}(W_l)$. These spaces W_l are finite dimensional over k. For the cases $l = 1, 2$ we have $\dim(W_1) = n - 1$ and $\dim(W_2) = \binom{n}{2}$, which are the dimensions of the Burau and Krammer representations, respectively. Hence, this gives an indication that ρ_1 and ρ_2 might reduce to ϕ_{r} and κ, respectively. And this is in fact the case since a direct calculation shows that $\left(\Delta^{(n)}(E)\right)(u_i) = 0$ and $\left(\Delta^{(n)}(E)\right)(w_{i,j}) = 0$, where $u_i = q^{\lambda}\hat{u}_i - \hat{u}_{i+1}$ and $w_{i,j} = \hat{w}_{i,j} - \alpha_{i,j}\hat{w}_j - \beta_{i,j}\hat{w}_i$, as is given in (5.36) and (5.54). That is to say, the spaces W_1 defined in section 5.3 and W_2 defined in section 5.4 are precisely the highest weight spaces of this section.

CHAPTER 6
TANGLE REPRESENTATIONS

In this chapter we discuss categorical representations of tangles. Braid group representations, as well as representations of the semigroup St_n of n-string links, can then be obtained by specialization. In particular we will obtain a representation of St_n which generalizes the Burau representation of B_n.

6.1 The Tangle Category

Recall that a geometric braid on n strands is, up to isotopy, a collection of n arcs A_i disjointly embedded in $D \times I$ such that $A_i(t) \in D \times \{t\}$, $A_i(0) = (z_i, 0)$, and $A_i(1) = (z_{\phi(i)}, 1)$ where $z_1, \dots z_n$ are n fixed points in D and $\phi \in S_n$. That is, a braid is made up of n disjoint paths A_i which move strictly upward and connect $(z_i, 0)$ to $(z_{\phi(i)}, 1)$. More generally, an n-string link is the same as a geometric braid except that we eliminate the condition $A_i(t) \in D \times \{t\}$.

Tangles are similar to string links in that they are made up of disjoint arcs in $D \times I$. For tangles, however, we do not require these arcs to join $D \times \{0\}$ to $D \times \{1\}$, but stipulate only that the endpoints of the arcs are somewhere in $\{z_1, \dots, z_m\} \times \{0, 1\}$. The definition which follows is more precise.

Let $\{z_1, z_2, \dots\}$ be a discrete collection of points in D. Let $\epsilon = (\epsilon_1, \dots, \epsilon_{2n})$ be a sequence of n $(+1)$'s and n (-1)'s in any order. We think of ϵ as being equal to the set $\{z_1, \dots, z_{2n}\}$ where each z_i is oriented by ϵ_i. Given any two oriented sets $\epsilon = (\epsilon_1, \dots, \epsilon_{2n})$ and $\epsilon' = (\epsilon'_1, \dots, \epsilon'_{2m})$ we can think of ϵ and ϵ' as being contained in

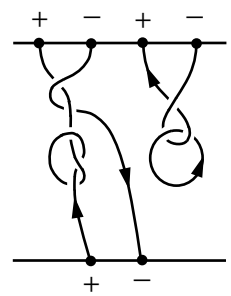


Figure 6.1: A generic tangle

$D \times I$ by $\epsilon \subseteq D \times \{0\}$ and $\epsilon' \subseteq D \times \{1\}$. Then ϵ and ϵ' can be joined by a *tangle*, by which we mean any collection of maps $a_i : I \to D \times I$, up to isotopy, such that exactly one of the following conditions holds (see Figure 6.1):

Condition 1. The point $a_i(0)$ is either a positive point of ϵ or a negative point of ϵ'; and the point $a_i(1)$ is either a negative point of ϵ or a positive point of ϵ'.

Condition 2. The map a_i has $a_i(0) = a_i(1)$ and lies completely in the interior of $D \times I$.

The *category of tangles* is the category $\mathcal{T}$ having objects the oriented sets $\epsilon = (\epsilon_1, \ldots, \epsilon_{2n})$ and morphisms all tangles from ϵ and ϵ'. The identity morphism in $\mathrm{Hom}_{\mathcal{T}}(\epsilon, \epsilon)$ is the tangle 1_ϵ which consists of vertical lines.

Note that if ϵ and ϵ' have the same number of points then they are isomorphic as objects of $\mathcal{T}$, an isomorphism between them being any "braid-like" tangle joining ϵ with ϵ'.

A *representation* of $\mathcal{T}$ is defined to be any functor F from $\mathcal{T}$ to the category k-$\mathcal{V}ect$ of vector spaces over the field k.

For any $n \in \mathbb{N}$ we define $\epsilon_n = (+1, \ldots, +1, -1, \ldots, -1)$ in $\mathrm{Obj}(\mathcal{T})$. We have $B_n \subseteq \mathrm{Aut}_{\mathcal{T}}(\epsilon_n)$ as shown in Figure 6.2 (a). Hence, any representation $F : \mathcal{T} \to k$-$\mathcal{V}ect$

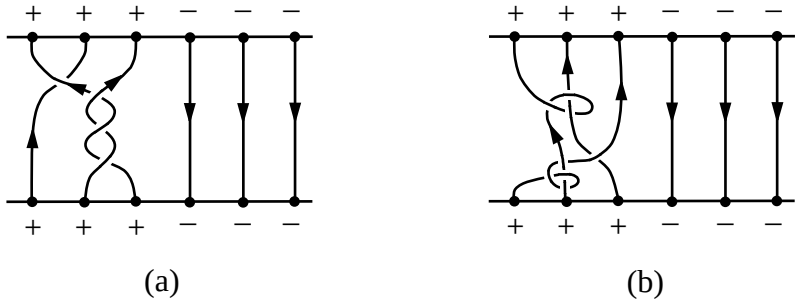


Figure 6.2: (a) Showing how braids are viewed in the tangle category. (b) Showing how string links are viewed in the tangle category.

of the tangle category restricts to an "honest" representation $B_n \to \mathrm{Aut}_k(F(\epsilon_n))$ of the braid groups.

Furthermore, the semigroup of n-string links St_n is contained in $\mathrm{End}_{\mathcal{T}}(\epsilon_n)$ as shown in Figure 6.2 (b). Hence, as for braids, a representation $F : \mathcal{T} \to k\text{-}\mathcal{V}ect$ of the tangle category will restrict to a representation $St_n \to \mathrm{End}_k(F(\epsilon_n))$ of the of the semigroup of n-string links.

6.2 An Algebra of Tangle Diagrams

Let $t \in \mathbb{C}$ be a nonzero complex number. For each $\epsilon = (\epsilon_1, \ldots, \epsilon_{2n})$ we define a complex vector space V_ϵ as follows.

We think of ϵ as sitting in $D \subseteq \mathbb{R}^3$. Define a *tangle diagram* s to be any collection of maps $a_i : I \to D \times (-\infty, 0] \subseteq \mathbb{R}^3$, up to isotopy, such that exactly one of the following conditions holds (see Figure 6.3 (a)):

Condition 1'. The point $a_i(0)$ is a negative point of ϵ and the point $a_i(1)$ is a positive point of ϵ.

Condition 2'. The map a_i has $a_i(0) = a_i(1)$ and lies completely in $D \times (-\infty, 0)$.

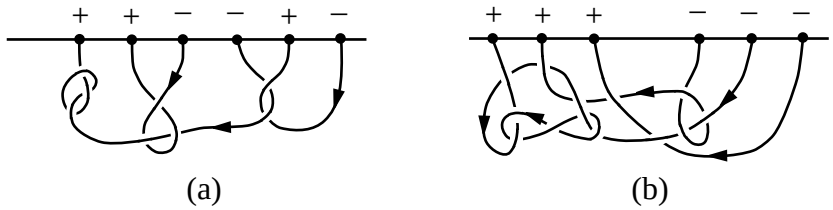
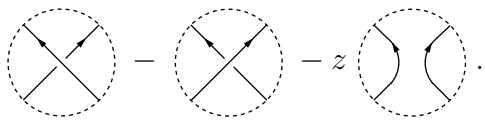


Figure 6.3: (a) A generic tangle diagram. Diagrams such as this one generate the space V_ϵ. (b) A tangle diagram in V_3.

Let W_ϵ be the free complex vector space generated by all tangle diagrams s. Let $R_\epsilon \subseteq W_\epsilon$ be the subspace generated by all elements of the form

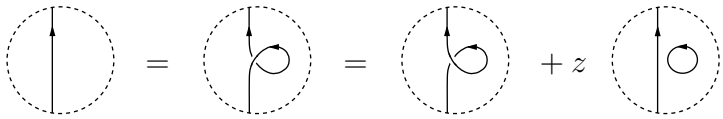


(6.1)

We define $V_\epsilon = W_\epsilon / R_\epsilon$ as a vector space over $\mathbb{C}$.

Equation (6.1) relates three tangle diagrams which are identical outside a small neighborhood of a crossing. That is, if s^+ is a tangle with a specified positive crossing, and if s^- and s^0 are the same tangle with the crossing changed to a negative and a smoothed crossing, respectively, then we have $s^+ = s^- + zs^0$ in V_ϵ.

Notice that tangle diagrams with a free component are zero in V_ϵ since by (6.1) any such diagram may be reduced to a combination of diagrams with a free circle and such diagrams are zero by



(6.2)

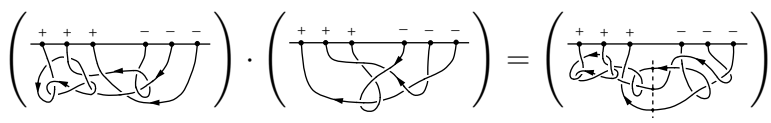
In this context, any tangle T in $\mathrm{Hom}_T(\epsilon, \epsilon')$ will naturally give a homomorphism $T_* : V_\epsilon \to V_{\epsilon'}$ which takes any tangle diagram s in V_ϵ to the diagram $T \cdot s$ in $V_{\epsilon'}$ obtained by stacking T on top of s. The homomorphism T_* is well defined since it preserves the relation $s^+ = s^- + zs^0$.

With these definitions in hand we now define a functor $F : \mathcal{T} \to k\text{-}\mathcal{V}ect$ by $F(\epsilon) = V_\epsilon$ and $F(T) = T_*$.

We would like to know more about these vector spaces V_ϵ. But as noted above, for any ϵ we have $\epsilon \cong \epsilon_n$ for some n. Hence, applying F we see that $V_\epsilon \cong V_{\epsilon_n}$.

Denote the space V_{ϵ_n} by V_n (an example of a tangle diagram in V_n is given in Figure 6.3 (b)). Suppose a tangle diagram s is given by maps $a_i : I \to D \times (\infty, 0]$ and we label the first n maps so that $a_i(0) = z_{2n-i+1}$. There is then an associated element $\pi(s) = (a_1(1), \ldots, a_n(1))$ of the symmetric group S_n. Let s_i denote the tangle diagram with $\pi(s_i) = (i\ i+1)$ and which has no crossings other than a single positive crossing. Also, let 1 denote the diagram that has no crossings.

We define a multiplication on V_n by concatenation. For example,



(6.3)

This multiplication is well defined since it preserves the relation $s^+ = s^- + zs^0$. Hence, V_n becomes a $\mathbb{C}$-algebra with this multiplication. Moreover, it is clear that V_n is finitely generated as an algebra over $\mathbb{C}$ by the elements $s_1, \ldots, s_{n-1}$ (and their inverses).

The diagrams s_i satisfy the braid relations

$$s_i s_j = s_j s_i \qquad |i - j| \geq 2 \tag{6.4}$$

$$s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \qquad 1 \leq i \leq n-2 \tag{6.5}$$

as well as the relation

$$s_i^2 = (t^{1/2} - t^{-1/2}) s_i + 1. \tag{6.6}$$

By setting $g_i = t^{1/2} s_i$ the relation in (6.6) becomes

$$g_i^2 = (t-1) g_i + t. \tag{6.7}$$

Hence, as a $\mathbb{C}$-algebra we have $V_n \cong H_n(t)$ where $H_n(t)$ is the Hecke algebra of type A_{n-1}. That is, $H_n(t)$ is the free complex algebra with generators $g_1, \ldots, g_{n-1}$ satisfying the braid relations and the relation given in (6.7).

The Hecke algebra $H_n(t)$ has been well studied. If t is not a root of unity it is isomorphic to $\mathbb{C}S_n$, the complex group algebra of the symmetric group, and its irreducible representations are indexed by Young diagrams (see [21] Theorem 2.2 and [9] Theorem 4.5; also, Cole Giller uses skein theory in [8] to give a direct proof that $V_n \cong \mathbb{C}S_n$).

B_n has a representation inside $H_n(t)$ by $\sigma_i \mapsto g_i$. Hence, any representation of $H_n(t)$ will restrict to a representation of B_n. Jones [9] initiated a detailed study of the representations of B_n which arise in this way. In particular he points out that by sending g_i to the matrix $-\psi_{\mathrm{r}}\sigma_i$ given in (3.2) one obtains an irreducible representation of $H_n(t)$. This is just a matter of verifying that $(-\psi_{\mathrm{r}}\sigma_i)^2 = (t-1)(-\psi_{\mathrm{r}}\sigma_i)+t$. Hence, by semisimplicity there is a submodule W of $H_n(t)$ such that the left regular action of $H_n(t)$ on W is given by the representation $g_i \mapsto -\psi_{\mathrm{r}}\sigma_i$.

Now the functor $F : \mathcal{T} \to k\text{-}\mathcal{V}ect$ defined by $F(\epsilon) = V_\epsilon$ and $F(T) = T_*$ is an extension to tangles of the left regular action of $H_n(t)$ on itself. That is, any tangle diagram $s \in V_n \cong H_n(t)$ can be straightend out and paired with n vertical lines, as in Figure 6.2, to give a tangle, i.e., a morphism in the category $\mathcal{T}$. Let T be this tangle. Then the action of T_* on V_n is precisely the left regular action of s on V_n.

Hence, by restricting to braids the functor defines a representation of the braid group on the $n-1$-dimensional module W by $\sigma_i \mapsto (\sigma_i)_*|_W$. The action of $(\sigma_i)_*$ on V_n is the same as the left regular action of s_i. We have $s_i = t^{-1/2}g_i$. Hence, the representation is given by $\sigma_i \mapsto -t^{-1/2}\psi_{\mathrm{r}}\sigma_i$. Of course, this is the Burau representation up to normalization by a constant.

Furthermore, restricting to string links gives a representation of St_n on W which generalzes the Burau representation. Matrices for this representation can be computed by using the decomposition rule $s^+ = s^- + zs^0$ to express string links as linear combinations of braids. We give an outline of why this should be the same as the generalized Burau representation of string links given by Lin [17].

The map $g_i \mapsto -1$ defines a one dimensional, hence irreducible, representaion of

the Hecke algebra. So there exists some submodule ε of $H_n(t)$ upon which each g_i acts by -1. We can then consider the the direct sum of these representations, $W \oplus \varepsilon$. It is given by

$$g_i \mapsto \left[\begin{array}{c|c} -\psi_{\mathrm{r}}\sigma_i & 0 \\ \hline 0 & \text{-1} \end{array}\right] \tag{6.8}$$

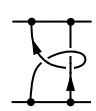
so that the induced representation on B_n is given by

$$\sigma_i \mapsto s_i = t^{-1/2} g_i \mapsto t^{-1/2} \left[\begin{array}{c|c} -\psi_{\mathrm{r}}\sigma_i & 0 \\ \hline 0 & \text{-1} \end{array}\right] = -t^{-1/2} \left[\begin{array}{c|c} \psi_{\mathrm{r}}\sigma_i & 0 \\ \hline 0 & 1 \end{array}\right]. \tag{6.9}$$

It is apparent from the equations in (3.2) that the unreduced Burau matrix, as given in (3.9), has eigenvector $(1/(t+1), 1, \ldots, 1)^T$ with eigenvalue 1. Hence, applying a basis transformation we see that this matrix is identical to the matrix in (6.9). Hence, the matrix in (6.9) can be put into the form that the underduced Burau matrix takes in equation (3.4). We may then take the transpose of this representation which is again a representation of the Hecke algebra. The matrices of this representation are *stochastic*, meaning that they have eigenvector $(1, \ldots, 1)^T$ with eigenvalue 1. We denote this representation by ψ.

We may then consider the representation ψ/ε, where by ε we now mean the map which takes g_i to -1, and therefore takes σ_i to $-t^{-1/2}$. This is well defined as a representation of string links. Moreover, the resulting matrix for a string link, say l, is a stochastic matrix since $\psi(l)$ has eigenvector $(1, \ldots, 1)^T$ with eigenvalue $\varepsilon(l)$. This is similar to what occurs in Lin's paper.

As an example, consider the string link l given by

$l =$  . (6.10)

This link decomposes as $l = 1 - z\sigma_1$. Hence, the matrix for l is given as follows

(cf. [17], page 295):

$$\frac{\psi(l)}{\varepsilon(l)} = \frac{1}{1 - z(-t^{-1/2})} \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - z(-t^{-1/2}) \begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix} \right] \tag{6.11}$$

$$= \frac{1}{2 - t^{-1}} \begin{pmatrix} 3 - t - t^{-1} & t - 1 \\ 1 - t^{-1} & 1 \end{pmatrix}. \tag{6.12}$$

Further, we note that this phenomena of extending the Burau representation to string links is not limited to just this particular representation. Indeed, *any* representation of the braid group which arises from an irreducible representation of the Hecke algebra will have an extension to St_n. The matrices for these representations are, again, computable by using the relation $s^+ = s^- + zs^0$.

6.3 The Conway Polynomial

The Conway polynomial is really just the Alexander polynomial normalized in such a way that we no longer have the abiguity concerning multiplication by a unit in $\mathbb{Z}[t, t^{-1}]$. The Conway polynomial is easy to compute, there being a skein relation by which we can reduce a link to a combination of unknots. We end this chapter by noting that the Conway polynomial arises out of the tangle theoretic context described in the previous section.

Proposition 18. *The Conway-normalized Alexander polynomial $\nabla_L(t)$ of a link L is a polynomial in $\mathbb{Z}[t^{1/2}, t^{-1/2}]$ which is characterized by*

$$\nabla_{unknot}(t) = 1 \tag{6.13}$$

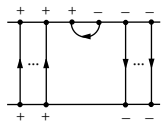
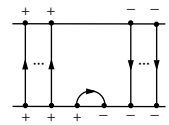
and

$$\nabla_{L^+} = \nabla_{L^-} + z\nabla_{L^0} \tag{6.14}$$

where $z = t^{1/2} - t^{-1/2}$ and the links L^+, L^-, and L^0 are identical outside a neighborhood of a crossing and differ inside the neighborhood according to equation (6.1).

Proof. See Lickorish [16], Theorem 8.6. □

Define tangles $T_i^+ : \epsilon_i \to \epsilon_{i+1}$ and $T_i^- : \epsilon_{i+1} \to \epsilon_i$ by

$T_i^+ =$  $T_i^- =$  (6.15)

Let β be a braid in B_n. Consider β to belong to $\mathrm{Aut}_{\mathcal{T}}(\epsilon_n)$ as in Figure 6.2 (a). The space V_1 is one-dimensional over $\mathbb{C}$ with basis given by the tangle diagram consisting of a single unknotted loop from z_2 to z_1. Hence, any linear map from V_1 to itself is multiplication by a constant in $\mathbb{C}$. Consider the map $(\nabla\beta)_*$ from V_1 to itself induced by the tangle

$$\nabla\beta = T_1^+ \dots T_{n-1}^+ \beta T_{n-1}^- \dots T_1^-. \tag{6.16}$$

Since the relation $s^+ = s^- + zs^0$ inside V_1 is identical to the skein relation in (6.14) we see easily the following identity:

$$(\nabla\beta)_* = \nabla_{\hat{\beta}}(t) \tag{6.17}$$

This should come as no surprise, though, given the similarity of the relations used define the Conway polynomial and the vector space V_n. Indeed, the Conway polynomial was virtually built into our definiton of the space V_n.

BIBLIOGRAPHY

[1] Emil Artin, *Theorie der Zöpfe*, Hamburg Abh., **4** (1925), 47-72.

[2] Stephen Bigelow, *The Burau representation is not faithfull for* $n = 5$, Geom. Topol., **3** (1999), 397–404.

[3] Stephen Bigelow, *Braid groups are linear*, J. Amer. Math. Soc., **14** (2001) no. 2, 471–486.

[4] Joan S. Birman, *Briads, links, and mapping class groups*, Princeton University Press (1974), Annals of Mathematics Studies, no. 82.

[5] W. Burau, *Über Zopfgruppen und gleichsinnig verdrillte Verkettungen*, Abh. Math. Semin. Hamburg. Univ., **11** (1935), 179–186.

[6] Gerhard Burde and Heiner Zieschang, *Knots*, de Gruyter (1985), de Gruyter Studies in Mathematics, no. 5.

[7] G. Felder and C. Wieczerkowski, *Topological Representations of the Quantum Group* $U_q(\mathfrak{sl}_2)$, Commun. Math. Phys., **138** (1991), 583–605.

[8] Cole A. Giller, *A Family of Links and the Conway Calculus*, Trans. Amer. Math. Soc., **270** (1982), no. 1, 75–109.

[9] V.F.R. Jones, *Hecke Algebra Representations of Braid Groups and Link Polynomials*, Ann. of Math., **126** (1987), no. 2, 335–388.

[10] Jurge Frolich and Thomas Kerler, *Quantum Groups, Quantum Categories and Quantum Field Theory*, Springer-Verlag, Berlin (1993), Lecture Notes in Mathematics, no. 1542.

[11] Christian Kassel, *Quantum Groups*, Springer-Verlag (1995).

[12] Christian Kassel, Marc Rosso, Vladimir Turaev, *Quantum Groups and Knot Invariants*, Societe Mathematique de France (1997), Panoramas et Syntheses, no. 5.

[13] Daan Krammer, *The braid group* B_4 *is linear*, Invent. Math., **132** (2000), no. 3, 451–486.

[14] Daan Krammer, *Braid Groups are linear*, preprint, Basel (2000).

[15] R.J. Lawrence, *Homological Representations of the Hecke Algebra*, Commun. Math. Phys., **135** (1990), 141–191.

[16] W.B. Lickorish, *An Introduction to Knot Theory*, Springer-Verlag (1997).

[17] Xiao-Song Lin, Feng Tian, and Zhenghan Wang, *Burau representation and rendom walks on string links*, Pacific J. Math., **182** (1998), no. 2, 289–302.

[18] John Moody, *The Burau representation of the braid group B_n is unfaithful for large n*, Bull. Amer. Math. Soc. (N.S.), **25** (1991), no. 2, 379–384.

[19] Daniel S. Silver and Susan G. Williams, *A generalized Burau representation for string links*, Pacific J. Math., **197** (2001), no. 1, 241–255.

[20] Vladimir Turaev, *Faithful Linear Representations of the Braid Groups*, arXiv: math.GT/0006202 (2000).

[21] Hanz Wenzl, *Hecke algebras of type A_n and subfactors*, Invent. Math., **92** (1988), no. 2, 349–383.

[22] Matthew Zinno, *On Krammer's Representation of the Braid Group*, arXiv:math. RT/0002136 (2000).