BETHE ANSATZ FOR CENTRAL ARRANGEMENTS OF HYPERPLANES

CRAIG JACKSON

1. Definitions

This is a note on the Bethe ansatz for central arrangments of hyperplanes. It follows a paper by Varchenko [2] that culminates in a proof of theorem 1. All the relavant definitions can be found in [2] and [1].

Let $\mathcal{C} = \{H_j\}$ be an arrangement of affine hyperplanes in $\mathbb{C}^k$. The arrangment $\mathcal{C}$ is *essential* if some subcollection of hyperplanes intersects in a point. The arrangement is *central* if $\bigcap_j H_j \neq \emptyset$. Let $f_j = 0$ be a defining equation for H_j and let $a : \mathcal{C} \to \mathbb{C}$ be a set of exponents for the arrangement. The function

$$\Phi_a = \prod_j f_j^{a(H_j)}$$

is the master function. This is a multi-valued function defined on $U = \mathbb{C}^k - \bigcup_j H_j$. A related function is the Hessian, which is given by

$$\mathrm{Hess}_a = \det(\frac{\partial^2}{\partial t_i \partial t_j} \ln \Phi_a)_{i,j}.$$

The Hessian is a rational function which is regular on U.

The Orlik-Solomon algebra is the graded algebra $\mathcal{A}(\mathcal{C})$ generated by the logarithmic differential forms $\omega_j = df_j/f_j$ on $\mathbb{C}^k$. It is equiped with a differential $d_a : \mathcal{A}^p(\mathcal{C}) \to \mathcal{A}^{p+1}(\mathcal{C})$ defined by $d_a : x \mapsto \omega_a \wedge x$, where $\omega_a = \sum_j a(H_j)\omega_j$. The space of flags $\mathcal{F}(\mathcal{C})$ is dual to $\mathcal{A}(\mathcal{C})$ via a pairing

$$\langle \, , \, \rangle : \mathcal{A}(\mathcal{C}) \otimes \mathcal{F}(\mathcal{C}) \to \mathbb{C}$$

and $\mathcal{F}(\mathcal{C})$ is equiped with the codifferential δ_a which is adjoint to d_a. An element $v \in \mathcal{F}^k(\mathcal{C})$ is said to be singular if it is in the kernal of δ_a. The space of singular elements is denoted by $\mathrm{Sing}\,\mathcal{F}^k(\mathcal{C})$.

A top degree form in $\mathcal{A}^k(\mathcal{C})$ can be written as

$$\eta = u \, dt_1 \wedge \cdots \wedge dt_k$$

for some rational function u which is regular on U. We define a specialization map $v : \mathbb{C}^k \to \mathcal{F}^k(\mathcal{C})$ by the formula

$$\langle \eta, v(t) \rangle = u(t) \qquad \text{for any} \qquad \eta \in \mathcal{A}^k(\mathcal{C}).$$

There is a symmetric bilinear form S_a defined on $\mathcal{F}(\mathcal{C})$. It is called the Shapovalov form.

2. Results

Varchenko [2] proves the following theorem:

Theorem 1. = *If $\mathcal{C}$ is an essential arrangment, then we have the following:*

(1) *A point $t \in U$ is a critical point of Φ_a if and only if $v(t)$ is singular.*
(2) *If $t \in U$, then*

$$S_a(v(t), v(t)) = (-1)^k Hess_a(t).$$

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(3) *If $t^1, t^2 \in U$ are different isolated critical points of Φ_a, then*

$$S_a(v(t^1), v(t^2)) = 0.$$

In order to apply this theorem, however, it would be nice to know if any critical points actually exist. A theorem by Orlik and Terao [1] helps here.

Theorem 2. *Let $\mathcal{C}$ be an essential complex arrangment with $|\mathcal{C}| = n$. There exists a (Zariski-) closed algebraic proper subset Y of $\mathbb{C}^n$ such that for each $a \in \mathbb{C}^n - Y$, Φ_a has only finitely many critical points all of which are nondegenerate. The number of critical points is independent of $a \in \mathbb{C}^n - Y$ and is equal to $|\chi(U)|$.*

Thus, 1 tells us that isolated critical points of Φ_a give independent singular values, while 2 tells us how to go about looking for critical points. Taken together these results give:

Theorem 3. *For generic exponents (i.e. $a \in \mathbb{C}^n - Y$) the set $\{v(t) | t \text{ is a critical point of } \Phi_a\}$ spans a $|\chi(U)|$-dimensional subspace in $Sing\mathcal{F}^k((C))$.*

Unfortunately, this does not tell us much in the case of a central arrangement. Indeed, $|\chi(U)| = 0$ for central arrangements. More specifically, the function Φ_a has critical points only when $\sum_j a(H_j) = 0$, and these exponents are explicitly excluded from the class of "generic" exponents defined in [1].

Hence the point of this note: to clarify the situation for central arrangments.

Assume from now on that $\mathcal{C}$ is central and the exponent a satisfies $\sum_j a(H_j) = 0$. Then the polynomials f_j are linear and we can choose coordinates $\{t_0, t_1, \ldots, t_{k-1}\}$ so that $f_0 = t_0$. Let K be the affine hyperplane defined by $t_0 = 1$. Then $\mathcal{C}$ defines a (generally noncentral) arrangment $\mathcal{C}' = \{h'_j\}_{j \neq 0}$ on K by $H'_j = H_j \cap K$. The defining polynomial of H'_j is $f'_j = f_j|_{t_0=1}$. We also associate to this arrangement the obvious exponent a' and we denote the master function of $\mathcal{C}'$ by $\Phi_{a'}$. The inclusion $i : K \hookrightarrow \mathbb{C}^k$ induces a degree preserving map $i^* : \mathcal{A}(\mathcal{C}) \to \mathcal{A}(\mathcal{C}')$. There is also a map $\mu : \mathcal{A}(\mathcal{C}') \to \mathcal{A}(\mathcal{C})$ defined by $x \mapsto \omega_0 \wedge x$. This map increases the degree by 1.

These maps fit into a split exact sequence

$$0 \longrightarrow \mathcal{A}(\mathcal{C}') \xrightarrow{\mu} \mathcal{A}(\mathcal{C}) \xrightarrow{i^*} \mathcal{A}(\mathcal{C}') \longrightarrow 0$$

which gives a dual splt exact sequence

$$0 \longrightarrow \mathcal{F}(\mathcal{C}') \xrightarrow{(i^*)^\top} \mathcal{F}(\mathcal{C}) \xrightarrow{\mu^\top} \mathcal{F}(\mathcal{C}') \longrightarrow 0.$$

Let $\alpha : \mathcal{F}(\mathcal{C}') \to \mathcal{F}(\mathcal{C})$ be a splitting of this exact sequence over $\mu^\top$. Then

$$\mathcal{F}(\mathcal{C}) = \mathrm{im}((i^*)^\top) \oplus \mathrm{im}(\alpha) \cong \mathcal{F}(\mathcal{C}') \oplus \mathcal{F}(\mathcal{C}').$$

Theorem 4. *Suppose $\mathcal{C}$ is central with $\sum_j a(H_j) = 0$, then we have the following:*

(1) *The point $t' \in K$ is a critical point of $\Phi_{a'}$ if and only if $\mathbb{C}^* t'$ is a critical set of Φ_a.*

(2) *The spaces $im((i^*)^\top)$ and $im(\alpha)$ are perpindicular to eachother: $S_a(im((i^*)^\top), im(\alpha)) = 0$. Moreover, for $x^1, x^2 \in \mathcal{F}(\mathcal{C}')$ we have $S_a(\alpha x^1, \alpha x^2) = a(H_0) S_{a'}(x^1, x^2)$ and $S_a((i^*)^\top x^1, (i^*)^\top x^2) = S_{a'}(x^1, x^2)$.*

References

[1] Peter Orlik, Hiroaki Terao, *The Number of Critical Points of Powers of Linear Functions*, Invent. math. 120, 1-14 (1995).

[2] Alexander Varchenko, *Bethe Ansatz for Arrangements of Hyperplanes and the Gaudin Model*, arXiv:math.QA/0408001 v4 (2004).