Specializations of the Lawrence Representations of the Braid Groups at Roots of Unity

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Abstract

We investigate the specialization $\mathtt{t}$ a primitive root of unity, ($\Phi_\ell(\mathtt{t}) = 0$), of the Lawrence representations $H_{n,\ell}$ of the braid groups B_n. We prove that under this specialization the Lawrence representation $H_{n,\ell}$ admits a subrepresentation isomorphic to the reduced Burau representation $\mathscr{B}_n$. Furthermore, we prove that the corresponding short exact sequence does not split for $n \geq 3$. We also show that a diagonal subrepresentation of the Lawrence representations, $H_{n,\ell}$, are isomorphic to a family of braid group representations derived from the quantum algebra $U_q(\mathfrak{sl}_2)$. This last result is an analog to a result recently proven by Ekenta and Jackson.

Contents

1 Introduction

The braid groups, B_n, were first defined by Emil Artin [1] in 1925 and since then have come to play an important role in many areas of mathematics and physics including topology, geometric group theory, quantum algebras, and conformal field theories. Representations of braid groups are important because they provide a concrete way to think about braid elements in terms of invertible matrices.

One of the longest standing open questions related to braid groups is whether or not they are linear. In other words, does there exists a faithful representation of B_n into a group of invertible matrices over a commutative ring. The earliest candidate for faithfulness was the Burau representation $\mathscr{B}_n$ discovered by Burau in 1935 [4]. In fact, it can be shown that $\mathscr{B}_n$ is faithful for $n \leq 3$. However, Bigelow proved that the Burau representation is not faithful for $n \geq 5$ [2]. The faithfulness of $\mathscr{B}_4$ is still unknown.

The question about the linearity of braid groups was answered in 2000 by Bigelow and Krammer [3, 8], who proved that the braid groups are linear by exhibiting a faithful representation of B_n on the homology module of a certain configuration space over the ring $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm 1}]$. This representation, known as the Lawrence-Krammer-Bigelow (LKB) representation, is one of a family of homology representations, $H_{n,\ell}$, first discovered by Ruth Lawrence in 1990 [9]. In particular, $H_{n,2}$ is the LKB representation while $H_{n,1}$ is isomorphic to the Burau representation $\mathscr{B}_n$. These results have rekindled an interest in the representation theory of braid groups.

In 2011, Jackson and Kerler proved that the quantum representations derived from the quantum algebra $U_q(\mathfrak{sl}_2)$ are irreducible over the quotient field $\mathbb{Q}(\mathtt{q}, \mathtt{t})$. Moreover, they showed that when the parameters are specialized to $\mathtt{tq} = -1$, the quantum representations admit a subrepresentation isomorphic to the Temperley-Lieb representation and that the the natural short exact sequence corresponding to this subrepresentation does not split for $n \geq 4$. Jackson and Kerler also showed that the LKB representation $H_{n,2}$ is isomorphic to a highest weight representation $W_{n,2}$ constructed from the quantum algebra $U_q(\mathfrak{sl}_2)$. Ekenta and Jackson subsequently extended this argument for all $\ell \geq 2$ by showing that $H_{n,\ell}$ is isomorphic to a diagonal subrepresentation of $W_{n,\ell}$ [5].

In our study, we investigate a different specialization of parameters. In particular, we show that when $\mathtt{t}$ is specialized at a primitive root of unity ($\Phi_\ell(\mathtt{t}) = 0$) the Lawrence representations admit a subrepresentation isomorphic to the Burau representation $\mathscr{B}_n$. We prove that the corresponding short exact sequence does not split for $n \geq 3$. Further, we demonstrate that an analog of the result by Ekenta and Jackson is similarly true. That is, we show that $W_{n,\ell}$ is isomorphic to a diagonal subrepresentation of $H_{n,\ell}$.

2 Group Theory

In this section, we define the notion of a group and give examples of important groups related to our discussion. We define the braid groups and state fundamental results concerning the braid groups. We leave out the proofs of these results and focus more on explaining and illustrating the concepts.

The concept of a group is central in abstract algebra, the study of algebraic structures. The definition of a group is general enough to capture the essential properties of number systems and many other familiar algebraic structures, yet specific enough to allow theories to be developed.

Definition 2.1. A **group** is a set G with a binary operation $*$ satisfying the following properties:

1. *Closure.* The operation $*$ assigns every ordered pair (a, b) of elements in G to an element $a * b$ in G. We say that G is *closed under* the operation $*$.

2. *Associativity.* The operation is associative; that is $(a * b) * c = a * (b * c)$ for all a, b, c in G.

3. *Identity.* There is an element e (called the *identity*) in G such that $a * e = e * a = a$ for all a in G.

4. *Inverse.* For each element a in G, there is an element, denoted a^{-1}, in G (called an *inverse* of a) such that $a * a^{-1} = a^{-1} * a = e$.

Groups are generalizations of number systems where elements in the group can be thought of as numbers and the binary operation as multiplication. This connection is motivated by the fact that the set of positive real numbers $\mathbb{R}^+$ together with multiplication is a group. It is worth mentioning that from this example we adopt the following practice for our convenience: we shall refer to the binary operation $*$ as multiplication even though the operation can be abstract. Consequently, we simply say that we *multiply* a and b in G as opposed to performing the operation $*$ on two elements a and b in G. If the context is clear, we shall refer to the group by the set itself.

We have stated the formal definition of a group and explained some group-related terms that we will use in the remainder of the paper. We have also introduced a basic example of a group, namely the set of positive real numbers $\mathbb{R}^+$. Next, we provide additional examples relevant to the discussion that follows.

Example 2.2. The set of $n \times n$ invertible matrices with entries in $\mathbb{R}$ and matrix multiplication forms the **general linear group**, denoted $GL(n, \mathbb{R})$.

The set of invertible matrices is closed under matrix multiplication. Since the set only contains invertible matrices, every matrix has an inverse. The identity element is the identity matrix I. The group of invertible matrices is important for many reasons which will be discussed in later sections. Elements in $GL(n, \mathbb{R})$ are nicely presented as tables of numbers. In addition, the operation defined on this group, namely matrix multiplication, is easy to compute. We also have a good understanding of this group based on the theory of linear algebra.

Another important example is the symmetric group S_n. First we introduce the concept of a **permutation** of a collection of n objects which we often think of as the set of n integers $\{1, \ldots, n\}$. Essentially, a permutation is a way to arrange these numbers in a different order. For example

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \tag{1}$$

is a permutation of the set $\{1, 2, 3, 4\}$ where we arrange the numbers so that 1 comes first then 3, 2 and 4. A formal way to think about a permutation is in terms of one-to-one functions from $\{1, \ldots, n\}$ to $\{1, \ldots, n\}$. In light of this view the permutation π in the previous example is a one-to-one function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3, 4\}$ so that

$$\pi(1) = 1, \quad \pi(2) = 3, \quad \pi(3) = 2, \quad \pi(4) = 4. \tag{2}$$

A binary operation on permutations can be defined by performing the permutations sequentially. The function interpretation of permutations allows us to define this binary operation as a much more familiar mathematical object, namely the composition of two functions. An example of multiplication of permutations as performed from right to left is

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 2 & 3 \end{pmatrix} \circ \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix} \tag{3}$$

Now that we have laid down the definitions of permutations and multiplication of two permutations we are ready to define the symmetric group.

Example 2.3. The set of permutations of an n-element set $\{1, \cdots, n\}$ with the operation of composition of permutation forms the **symmetric group** on n letters, denoted S_n.

The symmetric group is another fundamental group in abstract algebra. Every element in the group has an intuitive interpretation in terms of the ordering of n-objects. The symmetric group is

also easy to work with since multiplication of permutations is just function composition. Furthermore, the importance of the symmetric group is realized in Cayley's theorem. The theorem is stated in general terms as follows:

Theorem 2.4 (Cayley's Theorem)**.** *Every group is the same as (isomorphic to) a subgroup of a symmetric group.*

The implication of this theorem is that the study of group theory can be reduced to symmetric groups. Or, if we look at it the other way around, the study of groups provides us with a better understanding of the structure of the symmetric groups. Thus, this theorem highlights the fact that symmetric groups are an important and central concept of abstract algebra. Going forward we want to introduce a particular group of interest, namely the braid groups. Ultimately, it will become clear to us that braid groups are a generalization of the symmetric groups.

3 Braid Groups B_n

3.1 Geometric Braids

Braid groups were first introduced by Emil Artin in 1925. Since then they have played an important role in mathematics and physics. There are many different interpretations of braid groups which allow us to think about braid groups in different ways. We first appeal to intuition and provide a geometric interpretation of braid elements and braid composition. From there we will show that these geometric objects, as defined, satisfy the properties laid out in Definition 2.1.

Let us imagine n ants standing still at n starting points on a circular disk. These ants shall play a special edition of musical chairs. When the music starts, the ants move freely on the disk. When the music stops, the ants must come back to the starting points, but not necessarily their original starting point. Since this is a special edition of the game, all starting points will be kept on the disk so that the ants will always have a place to return to when the music stops. If we keep track of the paths of these ants and plot them with time on the vertical axis, the plot we obtain is a geometric representation of a **braid element**. The paths are referred to as the **strands** in the braid element. Figure 1 is a braid element on 4 strands. According to this braid, the ant starting at the third and second place ran around energetically and ended up switching places. Meanwhile, the weary ants starting at the first and fourth place only moved around their own place and eventually came back to it.

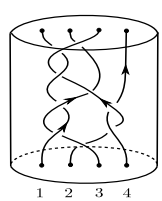


Figure 1: A braid element on 4 strands

Mathematically, the actual position of each strand in space and time is not as important as the winding pattern that these strands collectively form. Therefore we do not make mathematical distinction between braid elements with the strands stretched, shrunk, or moved around, as long as these strands do not pass through each other. Also, the size of the entire braid is not important. The composition (or multiplication) of braid elements is simply thought of as having these ants play two rounds of musical chairs consecutively. Geometrically, to multiply two braids we stack them on top of each other in the order of multiplication, connect the strands and remove the disk in between two braids. Figure 2 demonstrates the multiplication of two braids.

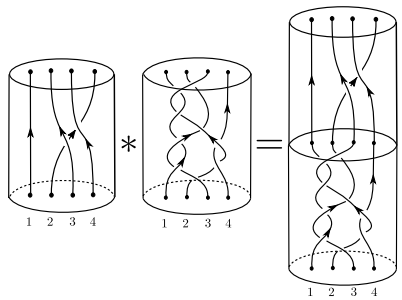


Figure 2: Braid multiplication

We see that by the definition of multiplication of braid elements the set of braids is closed under this operation. The identity element is simply the braid with all strands going straight up without winding around each other. Since the strands do not intertwine, the winding pattern of any other braid is unchanged when we multiply the two together. We shall leave it to the reader to verify that the inverse of a braid element can be obtained by reflecting the braid element across the bottom disk, and that braid multiplication is associative.

One can immediately realize that the braid groups are somewhat similar to the symmetric groups introduced in Example 2.3. The way n strands intertwine is related to the way n numbers are permutated. However, with braid groups we also keep track of how the strands intertwine in-between not just the beginning and final positions. Therefore, braid groups are a generalized version of the symmetric groups. This view will be important and become more clear when we introduce the symmetric representation of the braid groups.

3.2 Braid Generators and Relations

Certainly, the braid groups are more abstract than the previous groups we have encountered in this discussion. Braid groups have an infinite number of elements which are not the "usual numbers" we are familiar with. They are not built from numbers nor from familiar objects such as functions. So far we just have the geometric interpretation of braid elements. However, it would be more convenient if we had a different and maybe more concise definition of the braid groups. It turns out that the braid groups B_n can be defined in terms of **generators** and **relations**.

The way to look at groups in terms of generators and relations is somewhat analogous to the way we associate meanings to words. In English we have a lot of ways to write down the meaning "happy" by concatenating different letters from the alphabet in different ways such as "joyous" or "ecstatic". In mathematics, the elements of groups are the meanings that we can express while the generators form our alphabet. The relations tell us which strings of letters have the same meaning.

The following definition given by Emil Artin [1] defines braid groups in terms of generators and relations.

Definition 3.1. The **braid groups** B_n admit a presentation with generators $\sigma_1 \dots \sigma_{n-1}$ and relations

1. $\sigma_i\sigma_j = \sigma_j\sigma_i$, where $|i-j| \geq 2$,
2. $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ for all $1 \leq i \leq n-2$

where the element σ_i and σ_i^{-1} are the braid elements in Figure 3.

An example of how to express braid elements in terms of generators using the braid relations is that the braid element given in Figure 1 can be written as two different words, which are written from right to left:

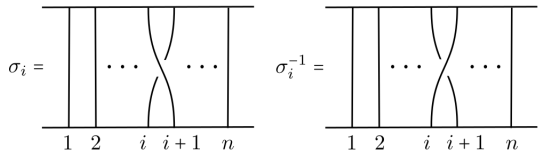


Figure 3: Braid elements σ_i and σ_i^{-1}

$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_1^{-1}\sigma_3\sigma_1^{-1}\sigma_2, \tag{4}$$

or equivalently

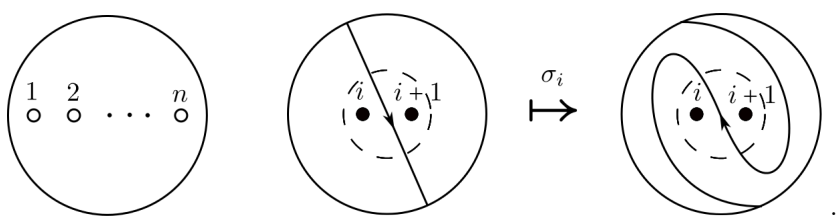
$$\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_3\sigma_2^{-1}\sigma_1^{-1}\sigma_1^{-1}\sigma_3\sigma_2. \tag{5}$$

In this example, the second word is obtained by applying the second relation on the product of the fourth and fifth letters of the first word, namely $\sigma_2\sigma_3$, and the first relation on the seventh and eighth letters of the first word, namely $\sigma_3\sigma_1^{-1}$. Although the two words are different, they both refer to the same braid element in B_n.

This definition provides a different and more concise way to think about braid groups. In general, if we want to think about the braid group B_n we only need to think about $n-1$ braid elements σ_i as shown in Figure 3. Specifically, in section 5 we shall see that to describe a representation of a braid group B_n we only need to give matrices corresponding to σ_i for each $i = 1, \ldots, n-1$. Furthermore, if we have a set of $n-1$ elements that obey the relations given in Definition 3.1, then they define a representation of B_n. We will return to this idea in our brief discussion on quantum representations of braid groups in section 9.1.

3.3 Dehn Half-Twists

Another important interpretation of the braid groups B_n is as a group of continuous mappings, namely homeomorphisms, on an n-punctured disk (Figure 4) that keep the boundary of the disk fixed. A proof of this fact can be found in [6]. From Definition 3.1, it suffices to define this group via the homeomorphisms corresponding to the generators σ_i. The homeomorphism corresponding to σ_i is the Dehn half-twist shown in Figure 4.



(a) n-punctured disk (b) Dehn half-twist σ_i

Figure 4

A mathematical description of the Dehn half-twist can be found in [6]. Geometrically, the Dehn half-twist is a transformation of the n-punctured disk to itself in which we lift the two punctures i and $i+1$ up, twist them in a clockwise direction and put them back into the disk. All Dehn half-twists fix the boundary. Furthermore, it can be shown that they satisfy the braid relations provided in Definition 3.1 and no others. Therefore, the Dehn half-twists can be viewed as generators of the braid groups B_n. The importance of this interpretation is that it provides us with an action of B_n on the n-punctured disk. This action, in turn, will allow us to define a representation of the braid groups. We will illustrate this point further in section 4.2.

4 Representation Theory

4.1 Definition

Groups can be abstract and complicated. The multiplication in an abstract group can be hard to carry out. Therefore, it may not always be convenient to work directly with an abstract group. One way we can study these abstract groups is via a function from the abstract group to other concrete groups. As we have seen in our examples earlier, groups of invertible matrices are easy to understand, so we use groups of invertible matrices as concrete groups in this situation. Since the structure of the group is the subject of the study, the function must preserve the group structure. In particular, it must behave nicely with respect to multiplications in both groups. To formalize these remarks, we provide the following definition of a representation.

Definition 4.1. A **representation** of a group G is a function from the group G to $GL(n,\mathbb{R})$, $\rho : G \to GL(n,\mathbb{R})$ such that for all g, h in G.

$$\rho(g * h) = \rho(g)\rho(h) \tag{6}$$

We call such function a **homomorphism**.

A representation is therefore just a way to correspond abstract elements with matrices. Equation 6 guarantees that we preserve the multiplicative structure of the abstract group G. In particular if we want to find the matrix corresponding to element $g * h$ for any g and h in G under the function ρ, we can either compute the product $g * h$ then apply function ρ, or we can apply function ρ to both g and h then multiply the two matrices. Thus, instead of doing the multiplication in G we can simply multiply matrices in $GL(n,\mathbb{R})$.

It is also worth noting that the representation may not preserve the entire group structure of G because we do not restrict the representation to just one-to-one functions. That is, the representation ρ can map many elements in G onto the same matrix in $GL(n,\mathbb{R})$ and still preserve the multiplication in G. Therefore, a matrix element does not provide enough information to determine one and only one element in the abstract group G. Precisely for this reason, the representation may only preserve some structure of the group G. In this case, we say the representation is **not faithful**. We shall see examples of this type of representation when we discuss the symmetric representation and the Burau representation of the braid groups. When the representation is one-to-one, we say that the representation is **faithful** and the group G is **linear**.

4.2 Representation via Group Action

As alluded to in the previous section, a representation of a group can be constructed from a group action. The concept of a group action is not at all unfamiliar in our discussion. In fact, there is always a group action lurking behind the examples of groups provided, namely the symmetric group S_n, the group of invertible matrices $GL(n,\mathbb{R})$ and the braid groups B_n. In the context of the symmetric group S_n, the action of S_n is the rearrangements of n numbers in the set $\{1,\ldots,n\}$, which we refer to as the space upon which S_n acts.

For the group of $n \times n$ invertible matrices $GL(n,\mathbb{R})$, the action is the familiar concept of invertible linear transformations acting on $\mathbb{R}^n$. For example, in $GL(2,\mathbb{R})$, the following matrix

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{7}$$

where θ is a number between 0 and 2π, is thought of as a linear transformation, namely the counter-clockwise rotation through an angle θ about the origin. Since these rotations, with the multiplication being the composition of functions, form a group, Equation 7 defines a representation of the group of

rotations on the plane. This provides us with an example of a representation of a group constructed via the action of the group on a space.

The braid groups B_n have a well-defined action on the n-punctured disk. Namely, each braid element corresponds to a homeomorphism on the n-punctured disk that fixes the boundary. In this correspondence, the generators σ_i are mapped to the Dehn half-twists. Furthermore, as we shall see in section 6, it is possible to associate a "vector space", precisely a module, to the n-punctured disk. Then, we can write down the action of the braid groups B_n on a basis of the module to obtain a representation of the braid groups B_n. A more detailed sketch of this procedure will be discussed in section 6.

4.3 Subrepresentation and Complementary subrepresetation

The last two crucial concepts of representation theory related to the project are the concepts of a **subrepresentation** and a **complementary subrepresentation**. We have seen that given a group G, a representation of G can be constructed via the action of the group on a certain vector space, say V. It is also possible that the group G can be represented via its action on a subspace W inside V. For example, it is possible to represent the counter-clockwise rotation R_θ through an angle θ about the origin in the three-dimensional vector space $\mathbb{R}^3$. In this case, the matrix representing the element R_θ is

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{8}$$

It should be clear from Equation 8 that Equation 7 defines a subrepresentation on a two dimensional subspace of $\mathbb{R}^3$. In terms of the coordinate system x, y and z, the subspace for the subrepresentation is the xy-plane living inside xyz-space. Note that this example is somewhat trivial since the action of R_θ on the z coordinate is trivial. In other words, R_θ does not change the z coordinate, and therefore the inclusion of the third dimension in the vector space for this representation is superfluous. However, this is not always the case. That is, it is possible to have a subrepresentation even when the group acts non-trivially on all vector components. More sophisticated examples of this kind will be introduced when we discuss the Burau representation and the specialization of the Lawrence representation.

We introduce some conventions and terminology. Since a representation depends on a choice of vector space and/or subspace on which the group acts, we refer to the representation by the name of the vector space. If there exists a subrepresentation U inside a representation W of the group G, we say that the representation W is **reducible**. Otherwise, we say that the representation W is **irreducible**. Furthermore, the existence of a subrepresentation U inside W allows us to form the following **short exact sequence**

$$0 \to U \overset{\iota}{\hookrightarrow} W \xrightarrow{\pi} W/U \to 0, \tag{9}$$

where W/U is the quotient representation formed by setting all elements in U to be zero in the group action of G on W. The sequence is called **exact** since the image of ι is the same as the kernel of π, which is the set of elements sent to 0 by π.

The concept of a subrepresentation provides a way to find new representation from known representations. In the trivial example concerning the two-dimensional rotations in the three-dimensional space, the subrepresentation arises naturally since the action of the rotations is fully contained inside the subspace, namely the xy-plane, in the three-dimensional space. Precisely, when we rotate any point in the xy-plane we obtain another point in the xy-plane. We say that a subspace is a subrepresentation if the action of the group on the subspace is **closed** with respect to the subspace.

Given a subrepresentation U inside W, a subspace V is **complementary** to U if and only if

$$W = U \oplus V \stackrel{\text{def}}{=} \{u + v \mid u \in U \text{ and } v \in V\} \qquad \text{and} \qquad U \cap V = \{0\}. \tag{10}$$

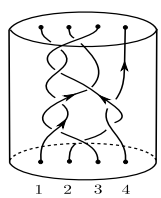
Furthermore, if the group action is closed on the complementary subspace V then we say that V is a **complementary subrepresentation**. In terms of the short exact sequence, we say that the short exact sequence **splits** if and only if a complementary subrepresentation exists.

5 Representations of Braid Groups

In this section, we introduce two classical representations of the braid groups, namely the **symmetric representation** and the **Burau representation**, of which the latter is related to our project. Finally, we introduce the **Lawrence representation** of the braid groups which is the main subject of our project.

5.1 The Symmetric Representation

The most natural and simple way to define a representation of the braid groups B_n is via the symmetric group S_n. Recall the musical chairs example given earlier: braid elements in B_n keep track of the trajectories of n ants. Suppose instead of keeping track of all trajectories, we only keep track of the initial and the final positions of the ants. Therefore under this new rule, one round of musical chairs corresponds to a permutation of n ants. This gives us a function from the braid group B_n to the group of permutations S_n. For example, the braid element in Figure 1 corresponds to the permutation given in Equation 1.



$$\longmapsto \quad \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \tag{11}$$

Since multiplication of permutations is similar to that of braid elements, the function behaves nicely with respect to multiplication of braid elements. In fact, if we compare examples of multiplication given in Equation 3 and Figure 2 we see that the permutation corresponding to the product of braid elements is the same as the product of the two corresponding permutations. We should note that there are many other braid elements corresponding to the same permutation. For example, the braid generator σ_2 and its inverse σ_2^{-1} are two of them.

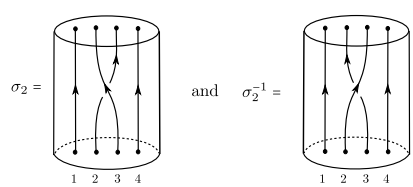


Figure 5: Braid elements σ_2 and σ_2^{-1}

Therefore, this function provides an unfaithful representation of the braid group B_n into the symmetric group S_n. Under this representation, the generators σ_i are mapped to the permutation that switches elements i and $i+1$ and fixes the rest. These permutations are called **transpositions**. As mentioned in the discussion for Definition 3.1, it suffices to define the representation of B_n on only the $n-1$ generators σ_i. Since these generators correspond to the transpositions mentioned above, we define the matrix representation of σ_i via the action of the corresponding transposition on the set

$\{1, \ldots, n\}$. The transposition acts on the set of numbers by exchanging i and $i+1$ while fixing other numbers. The natural linear transformation that corresponds to this action is the transformation from $\mathbb{R}^n$ to itself that exchanges the ith and $(i+1)^{st}$coordinates while fixing all other coordinates. Thus, the following proposition defines the symmetric representation of B_n.

Proposition 5.1. *The function $\rho : B_n \to GL(n, \mathbb{R})$ given by*

$$\rho(\sigma_i) = \begin{pmatrix} I & & & 0 \\ & 0 & 1 & \\ & 1 & 0 & \\ 0 & & & I \end{pmatrix}, \tag{12}$$

where the ith and $(i+1)^{st}$ columns of the identity matrix are switched, defines the ***symmetric representation*** *of B_n into $GL(n, \mathbb{R})$.*

Note that the when we multiply the matrix given above with a column vector, all entries of the vector remain the same except for the ith and $(i+1)^{st}$ elements, which are switched. Therefore, the matrices given above satisfiy the braid relations given in Definition 3.1. Thus the function defined is a representation of B_n.

5.2 The Burau Representation

A generalized version of the symmetric representation is the unreduced Burau representation defined in the following proposition.

Proposition 5.2. *The function $\rho : B_n \to GL(n, \mathbb{Z}[\mathtt{t}^{\pm 1}])$ given by*

$$\rho(\sigma_i) = \begin{pmatrix} I & & & 0 \\ & 1-\mathtt{t} & \mathtt{t} & \\ & 1 & 0 & \\ 0 & & & I \end{pmatrix}, \tag{13}$$

where the $1-\mathtt{t}$ term occurs at the (i,i) entry, defines the ***unreduced Burau representation*** *of B_n into $GL(n, \mathbb{Z}[\mathtt{t}^{\pm 1}])$.*

The unreduced Burau representation is a generalized version of the symmetric representation because with the specialization $t = 1$ we obtain the matrices for the symmetric representation. Furthermore, the unreduced Burau representation is reducible because there exists a subrepresentation. It can be further shown that the subrepresentation is irreducible, so we refer to it as the reduced Burau representation.

Proposition 5.3. *The function $\rho : B_n \to GL(n-1, \mathbb{Z}[\mathtt{t}^{\pm 1}])$ given by*

$$\rho(\sigma_i) = \begin{pmatrix} I & & & & 0 \\ & 1 & 0 & 0 & \\ & \mathtt{t} & -\mathtt{t} & 1 & \\ & 0 & 0 & 1 & \\ 0 & & & & I \end{pmatrix}, \tag{14}$$

where the top left 1 term occurs at the $(i-1, i-1)$ entry, defines the ***reduced Burau representation*** *of B_n into $GL(n-1, \mathbb{Z}[\mathtt{t}^{\pm 1}])$.*

For the rest of the paper, we refer to this reduced representation as the Burau representation, $\mathscr{B}_n$. For a long time, the Burau representation was thought to be a good candidate for faithfulness and thus answer the question about the linearity of braid group B_n. Indeed, a simple argument shows that the Burau representation of B_3 is faithful. However, Bigelow proved that the representation is not faithful for all $n \geq 5$. The faithfulness for the case $n = 4$ remains open.

6 The Lawrence Representations of the Braid Groups

The Lawrence representations are a family of two-parameter, $\mathtt{q}$ and $\mathtt{t}$, representations $H_{n,\ell}$, that were first constructed by Ruth Lawrence in 1990 [9]. The Lawrence representations $H_{n,\ell}$ contain familiar representations for small values of ℓ. The first member of the family of Lawrence representations, namely $H_{n,1}$, is isomorphic to the Burau representation $\mathscr{B}_n$. The second member of the family of Lawrence representations, $H_{n,2}$, is the Lawrence-Krammer-Bigelow (LKB) representation, which was shown to be faithful for all n by Krammer and Bigelow [3, 8]. The faithfulness of the LKB representation answered the open question about the linearity of the braid groups B_n. The faithfulness of the representations $H_{n,\ell}$ with $\ell \geq 3$ remains open.

In this section, we introduce a construction of the Lawrence representations $H_{n,\ell}$ via the braid action on a n-punctured disk. In particular, we briefly discuss the construction of the Lawrence representation for the case $\ell = 2$, namely the LKB representation $H_{n,2}$. The construction of the Lawrence representation $H_{n,\ell}$ is a generalization of the case $\ell = 2$. Most importantly, we provide a nice way to represent elements of $H_{n,\ell}$ in terms of forks.

6.1 Construction of the LKB Representation $H_{n,2}$

Let $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ be the unit disk in the complex plane, $P = \{p_1, \ldots, p_n\}$ be a set of n distinct points in D. Then, $D_n = D - P$ denotes the n-punctured disk on which the braid group B_n acts. We define C as the space of unordered pairs of points in D_n. In other words, C is the quotient space

$$C = \frac{(D_n \times D_n) - \{(x,x)\}}{(x,y) \sim (y,x)}. \tag{15}$$

Suppose d_1 and d_2 are distinct points on the boundary of the n-punctured disk D_n. Then by Equation 15, $c_0 = \{d_1, d_2\}$ is a point in C. Considering all closed loops in C that start and end at c_0 with the natural multiplication defined by the composition of loops, we obtain the fundamental group of C, denoted by $\pi_1(C, c_0)$. Let α be a closed loop in C. By Equation 15, the closed loop α in C can be thought of as an unordered pair of arcs α_1 and α_2 in the n-punctured disk D_n, $\alpha(s) = \{\alpha_1(s), \alpha_2(s)\}$. We define a map ϕ from the fundamental group $\pi_1(C, c_0)$ to the free, i.e. no relations, Abelian group with generators $\{\mathtt{q}, \mathtt{t}\}$ such that

$$\phi(\alpha) = \mathtt{q}^a \mathtt{t}^b, \tag{16}$$

where a is the number of times the loops α_1 and α_2 wind around the puncture points, and b is the number of times the loops α_1 and α_2 wind around each other.

Let $\tilde{C}$ be the regular covering of C corresponding to the kernel of ϕ and let $\tilde{c_0}$ be a lift of c_0 to $\tilde{C}$. The parameters $\mathtt{q}$ and $\mathtt{t}$ act on the covering space $\tilde{C}$ as deck transformations. The homology group $H_2(\tilde{C})$ becomes a $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm}]$-module. Thus, we have associated to the n-punctured disk D_n a $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm 1}]$-module, namely $H_{n,2} \stackrel{\text{def}}{=} H_2(\tilde{C})$, which is essentially a vector space.

The braid action on the n-punctured disk D_n naturally induces an action on the homology space. In particular, any braid action β in B_n induces a homeomorphism h on the n-punctured disk D_n that fixes the boundary of the disk. The homeomorphism h on D_n induces a homeomorphism $C \to C$ via the mapping $h : \{x, y\} \mapsto \{h(x), h(y)\}$. The map h lifts to a homeomorphism $\tilde{h} : \tilde{C} \to \tilde{C}$ which commutes with the deck transformations $\mathtt{q}$ and $\mathtt{t}$. This map $\tilde{h}$ induces a $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm}]$-module isomorphism $\tilde{h}_* : H_n(\tilde{C}) \to H_n(\tilde{C})$ that represents the braid action β on $H_n(\tilde{C})$ in B_n.

6.2 Forks as Elements of $H_{n,\ell}$

One way to represent elements of the module $H_{n,\ell}$ is to use forks as suggested by Krammer. We provide a description of forks in $H_{n,2}$ and the relations needed for fork decomposition. A fork in $H_{n,2}$ is a union of two embedded graphs in D_n as shown in Figure 6.

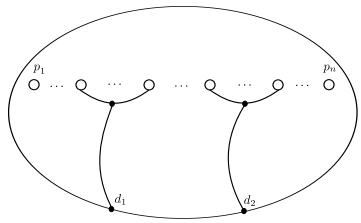


Figure 6: A fork representation of an element in $H_{n,2}$

We call the union of the two edges in each graph, which connect to two *distinct* punctures, the **tine edges** of the fork. We call the edges connecting the tine edges and the boundary the **handle** of the fork. We note that the tine edges can wind around the punctures as long as they do not intersect. In other words, a tine edge must not intersect a different tine edge.

In fact, the tine edges of a fork represent an equivalence class of forks that is unique up to some application of the deck transformations $\mathtt{q}$ and $\mathtt{t}$. Without going into too much detail, we provide the equivalence relations of forks in the equivalence class. For any fork, unwrapping a handle around a puncture results in a factor of $\mathtt{q}$ multiplied to the fork. In addition, unwrapping the handles around each other results in a factor of $\mathtt{t}$. The change in relative orientation between the tine edge and the handle leads to a change of sign. Figure 7 summarizes the equivalence relation on forks. Furthermore, a fork can be decomposed into a $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm 1}]$-linear combination of forks in $H_{n,\ell}$ as shown in Figure 8.

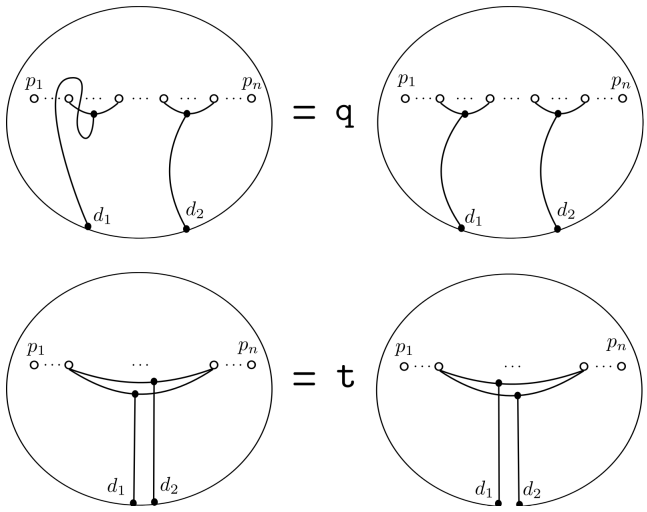


Figure 7: Equivalent relations between the forks in $H_{n,2}$.

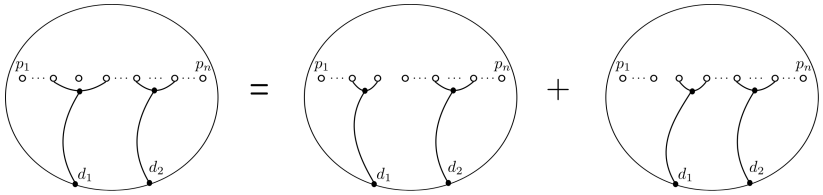


Figure 8: Fork decomposition

The usage of forks to represent elements in $H_{n,2}$ can be generalized to represent elements in $H_{n,\ell}$. In $H_{n,\ell}$, an element is represented by a fork with ℓ tine edges and ℓ handles which connect to ℓ points on the boundary of the n-punctured disk D_n. The equivalence relations between the forks in $H_{n,\ell}$ are similar to those in $H_{n,2}$.

7 Combinatorial Preliminaries

Before we start a discussion of the braid action on $H_{n,\ell}$, we shall define some useful combinatorial gadgets. These are so-called quantum numbers and related quantities: q-numbers, q-factorials, q-binomial coefficients, $\mathtt{t}$-numbers, $\mathtt{t}$-factorials, and $\mathtt{t}$-binomials coefficients. Although the connection between these expressions and the representation may not be obvious from their definitions, the $\mathtt{t}$-binomial coefficients, as we will see, appear frequently in the formulas for the Lawrence representation. In section 9.1, we will see the presence of the q-binomial coefficients in the formulas for certain "quantum" representations of the braid groups B_n.

Definition 7.1. We define $\mathtt{t}$-numbers, $\mathtt{t}$-factorials, and $\mathtt{t}$-binomial coefficients as follows

$$(n)_{\mathtt{t}} = \frac{\mathtt{t}^n - 1}{\mathtt{t} - 1} = 1 + \mathtt{t} + \cdots + \mathtt{t}^{n-1} \qquad (n)_{\mathtt{t}}! = (n)_{\mathtt{t}} \cdots (1)_{\mathtt{t}} \qquad \binom{n}{j}_{\mathtt{t}} = \frac{(n)_{\mathtt{t}}!}{(n-j)_{\mathtt{t}}!(j)_{\mathtt{t}}!} \tag{17}$$

The q-numbers, q-factorials, and q-binomial coefficients can also be defined in a similar way.

Definition 7.2. We define q-numbers, q-factorials, and q-binomial coefficients as follows

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \qquad [n]_q! = [n]_q \cdots [1]_q \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = \frac{[n]_q!}{[n-j]_q![j]_q!} \tag{18}$$

Note that the expressions for $\mathtt{t}$- and q- numbers are similar to each other. In particular, we can relate the $\mathtt{t}$- and q- numbers, factorials, and binomial coefficients in Definition 7.1 and 7.2 by the following equations

$$[n]_q = q^{-(n-1)}(n)_{q^2} \qquad [n]_q! = q^{-c_n}(n)_{q^2}! \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = q^{j(j-n)}\binom{n}{j}_{q^2} \tag{19}$$

where the c_n that appears in the second identity denotes the sum of the first n non-negative integers:

$$c_n \stackrel{\text{def}}{=} 0 + 1 + \cdots + (n-1) = \frac{n(n-1)}{2} \tag{20}$$

Equations 19 suggests that the $\mathtt{t}$- and q- numbers, factorials, and binomial coefficients are similar to each other. The similarity between these two combinatorial gadgets will allow us to introduce an identification of parameters and an isomorphism between the Lawrence and the quantum representations of braid groups. We will revisit this idea in section 9.1.

The analogy between these expressions and the natural numbers is revealed when we make the specialization $\mathtt{t} = 1$ and/or $q = 1$. In this specialization, we have $(n)_{\mathtt{t}} = [n]_q = n$. It also follows that in the specialization $\mathtt{t} = 1$ and/or $q = 1$, the definitions of the $\mathtt{t}$- and q- factorials and binomial coefficients coincide with the definitions of factorial and binomial coefficients defined on the natural numbers. From this analogy, the following identities are recognized as the $\mathtt{t}$- and q- analogs of familiar identities of the usual factorials and binomial coefficients.

Proposition 7.3.

$$(n+1)_{\mathtt{t}} = \mathtt{t}(n)_{\mathtt{t}} + 1 \qquad [n+1]_q = q[n]_q + q^{-n} \tag{21}$$

$$\binom{n}{j}_{\mathtt{t}} = \binom{n}{n-j}_{\mathtt{t}} \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = \begin{bmatrix} n \\ n-j \end{bmatrix}_q \tag{22}$$

$$\binom{n}{j}_{\mathtt{t}} = \mathtt{t}^j \binom{n-1}{j}_{\mathtt{t}} + \binom{n-1}{j-1}_{\mathtt{t}} \qquad \begin{bmatrix} n \\ j \end{bmatrix}_q = q^j \begin{bmatrix} n-1 \\ j \end{bmatrix}_q + q^{j-n} \begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_q \tag{23}$$

In our project, we are interested in the specialization $\Phi_\ell(\mathtt{t}) = 0$. Therefore, it is important to look at the combinatorial gadgets under this specialization. We notice that certain $\mathtt{t}$-binomial coefficients vanish in the specialization $\Phi_\ell(\mathtt{t}) = 0$. In particular, we have the following lemma.

Lemma 7.4. *For all $0 < k < \ell$, under the specialization $\mathtt{t}$ is a primitive ℓ^{th} root of unity, $\Phi_\ell(\mathtt{t}) = 0$, we have*

$$\binom{\ell}{k}_{\mathtt{t}} = \frac{(\ell)_{\mathtt{t}}!}{(\ell-k)_{\mathtt{t}}!(k)_{\mathtt{t}}!} = 0. \tag{24}$$

The proof of Lemma 7.4 is based on the factorization of the $\mathtt{t}$-number $(\ell)_{\mathtt{t}}$ in terms of certain cyclotomic polynomials.

Lemma 7.5. *For any natural number ℓ, the $\mathtt{t}$-number can be written as*

$$(\ell)_{\mathtt{t}} = 1 + \mathtt{t} + \cdots + \mathtt{t}^{\ell-1} = \prod_{\substack{1<d\leq\ell \\ d|\ell}} \Phi_d(\mathtt{t}) \tag{25}$$

where Φ_d is the dth cyclotomic polynomial defined as

$$\Phi_d(\mathtt{t}) \stackrel{\text{def}}{=} \prod_{\substack{1\leq j\leq d \\ \gcd(j,d)=1}} \left(\mathtt{t} - e^{2\pi j i/d}\right) \tag{26}$$

Proof of Lemma 7.4. We first notice that when $k = 0$ or $k = \ell$, the $\mathtt{t}$-binomial coefficient is 1 regardless of the specialization. To prove that the binomial coefficients vanish for other values of k, we show that at certain roots of unity the numerator vanishes while the denominator is non-zero. Using Lemma 7.5, we factor the numerator and the denominator of the $\mathtt{t}$-binomial coefficient into products of cyclotomic polynomials as follows

$$(\ell)_{\mathtt{t}}! = \prod_{j=2}^{\ell} \prod_{\substack{1<d\leq j \\ d|j}} \Phi_d(\mathtt{t}) \qquad \text{and} \qquad (\ell-k)_{\mathtt{t}}!(k)_{\mathtt{t}}! = \left(\prod_{j=2}^{\ell-k} \prod_{\substack{1<d\leq j \\ d|j}} \Phi_d(\mathtt{t})\right)\left(\prod_{j=2}^{k} \prod_{\substack{1<d\leq j \\ d|j}} \Phi_d(\mathtt{t})\right). \tag{27}$$

Now, we observe that for any $0 < k < \ell$ the factor $\Phi_\ell(\mathtt{t})$ only appears in the numerator but not the denominator. Furthermore, the roots of $\Phi_\ell(\mathtt{t})$ are the ℓth primitive roots of unity. Therefore, the $\mathtt{t}$-binomial coeffiecient $\binom{\ell}{k}_{\mathtt{t}}$ vanishes under the specialization of $\mathtt{t}$ a primitive ℓ^{th} root of unity for any $0 < k < \ell$. □

8 Specialization of the Lawrence Representations

In this section, we discuss the specialization $\Phi_\ell(\mathtt{t}) = 0$ of the family of Lawrence representations, $H_{n,\ell}$ of the braid groups B_n. The Lawrence representation is irreducible over the ring $\mathbb{Z}[\mathtt{t}^{\pm1}, \mathtt{q}^{\pm1}]$. However, we show that they are reducible under the specialization $\mathtt{t}^\ell = 1$. We prove that a subrepresentation, isomorphic to the Burau representation, can be recovered from the Lawrence representation $H_{n,\ell}$ under the specialization $\mathtt{t}^\ell = 1$ for all $n \geq 2$ and $\ell \geq 0$. Specifically, we find

Theorem 8.1. *For all $n \geq 2$, $\ell \geq 0$, the B_n-representation $H_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$ has a subrepresentation isomorphic to the Burau representation, $\mathscr{B}_n$.*

Furthermore, we prove that there is no complementary sub-representation to the Burau representation in $H_{n,\ell}$ under the specialization. In other words,

Theorem 8.2. *The following short exact sequence does not split for all $n \geq 2$ and $\ell \geq 0$,*

$$0 \to \mathscr{B}_n \hookrightarrow H_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0} \to H_{n,\ell}/\mathscr{B}_n|_{\Phi_\ell(\mathtt{t})=0} \to 0. \tag{28}$$

To prove theorem 8.1, we consider the action of the braid group B_n on a submodule of $H_{n,\ell}$, namely $A_{n,\ell}$, spanned by a basis $\mathcal{A}_{n,\ell}$. The basis $\mathcal{A}_{n,\ell}$ is defined as a subset of the basis $\mathcal{H}_{n,\ell}$ of $H_{n,\ell}$. The basis $\mathcal{H}_{n,\ell}$ consists of the fork elements with e_i tine edges between puncture i and $i+1$ for all $1 \le i \le n-1$ such that the total number of tine edges is $e_1 + \cdots + e_{n-1} = \ell$. The handles of these forks, from left to right, are connected to the corresponding tine edges from top to bottom. This order is referred to as the **standard order** of the handles, and the forks in $\mathcal{H}_{n,\ell}$ are the **standard forks**. Formally, we define $\mathcal{H}_{n,\ell}$ as

$$\mathcal{H}_{n,\ell} = \left\{ F_{\mathbf{e}} \;\middle|\; \mathbf{e} = (e_1, \ldots, e_{n-1}), \sum_{i=1}^{n-1} e_i = \ell, \text{ and } 0 \le e_i \le \ell \right\}, \tag{29}$$

where $F_{\mathbf{e}}$ will be referred to as the standard fork. In the following sections, we will define the basis, $\mathcal{A}_{n,\ell}$, of the submodule $A_{n,\ell}$.

8.1 Braid Actions of σ_i on $A_{n,\ell}$

Let $\mathbf{e}_i = (0, \ldots, 0, \ell, 0, \ldots 0)$ where ℓ occurs at the ith entry of $\mathbf{e}_i$. We define $A_{n,\ell}$ as a submodule of $H_{n,\ell}$ spanned by $\mathcal{A}_{n,\ell} = \{F_{\mathbf{e}_i} | 1 \le i \le n-1\}$ where $F_{\mathbf{e}_i}$ represents the fork with all ℓ tine edges connecting i^{th} and $(i+1)^{st}$ punctures as shown in Figure 9.

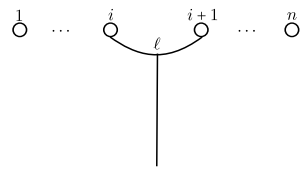


Figure 9: The fork $F_{\mathbf{e}_i}$

We will let the braid element σ_i act on the fork basis $\mathcal{A}_{n,\ell}$ of $A_{n,\ell}$ and decompose the result into the fork basis of $H_{n,\ell}$. We notice that the braid generator σ_i does nothing to the fork $F_{\mathbf{e}_j}$ when $|i-j| > 1$. Therefore, we have

$$\sigma_i F_{\mathbf{e}_j} = F_{\mathbf{e}_j} \qquad |i-j| > 1. \tag{30}$$

In other words, the action of σ_i only affects tine edges connecting to either the ith or the $(i+1)^{st}$ puncture. Therefore, we need to consider three cases: the action of σ_i on $F_{\mathbf{e}_{i-1}}$, $F_{\mathbf{e}_i}$, and $F_{\mathbf{e}_{i+1}}$. For our convenience, we introduce the **local forks** around the ith and $(i+1)^{st}$ punctures in which we simply ignore the tine edges connecting punctures outside the range from i to $i+2$. The action of σ_i can be generalized from the action of σ_i on these local forks.

8.2 Local Forks

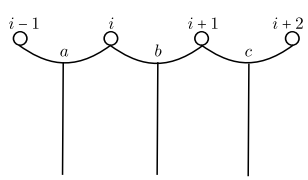
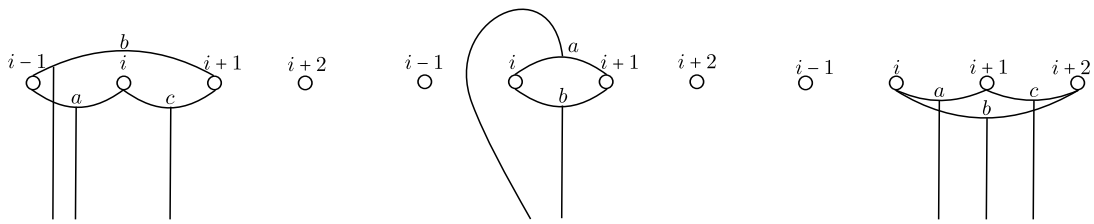


Figure 10: $F_{a,b,c}$

Let i be fixed. We define the local fork of type F, denoted by $F_{a,b,c}$, to be the fork in Figure 10. The action of σ_i on $F_{\mathbf{e}_{i-1}}$, $F_{\mathbf{e}_i}$, and $F_{\mathbf{e}_{i+1}}$ can be generalized from the action of σ_i on $F_{\ell,0,0}$, $F_{0,\ell,0}$, and $F_{0,0,\ell}$ since in this notation we have

$$F_{\mathbf{e}_{i-1}} = F_{\ell,0,0} \qquad F_{\mathbf{e}_i} = F_{0,\ell,0} \qquad F_{\mathbf{e}_{i+1}} = F_{0,0,\ell}. \tag{31}$$

Furthermore we define local forks of types A, B, and C, namely $A_{a,b,c}$, $B_{a,b}$ and $C_{a,b,c}$ to be the forks shown in Figure 11.



(a) $A_{a,b,c}$ for $2 \le i \le n-1$ (b) $B_{a,b}$ for $1 \le i \le n-1$ (c) $C_{a,b,c}$ for $1 \le i \le n-2$

Figure 11: Local forks of types A, B, and C.

The different ranges of index i for each type of fork makes sure that our definition is well-defined for each fork. We observe that by this definition the action of σ_i on $F_{\ell,0,0}$, $F_{0,\ell,0}$, and $F_{0,0,\ell}$ yields the following

$$\sigma_i F_{\ell,0,0} = A_{0,\ell,0} \qquad \sigma_i F_{0,\ell,0} = B_{\ell,0} \qquad \sigma_i F_{0,0,\ell} = C_{0,\ell,0}. \tag{32}$$

Our remaining task in this section is to decompose these forks $A_{0,\ell,0}$, $B_{\ell,0}$ and $C_{0,\ell,0}$ into standard forks in $H_{n,\ell}$. We shall do this in general for forks $A_{a,b,c}$, $B_{a,b}$ and $C_{a,b,c}$ and restrict our attention to $A_{0,\ell,0}$, $B_{\ell,0}$ and $C_{0,\ell,0}$.

8.3 Action of σ_i on Local Forks

The decomposition of $B_{a,b}$ is straight-forward. The fork $B_{a,b}$ has a number, a, of its tine edges with their handles wrapping round the ith puncture. All we need to do to turn $B_{a,b}$ to a standard fork is to move all a handles through the ith puncture and rearrange them in the standard order. When we move all a handles through the ith puncture, we get a factor of $\mathtt{q}^a$. We reorient these a handles to connect the tine edges from below. This gives us a factor of $(-1)^a$. Now notice that all a handles are in the opposite order as compared to the standard order. Therefore, we need to pull each handle, starting on the left, through all the handles on the right. Starting from the left-most handles, we would get a factor of $\mathtt{t}^{a-1}$; for the next handles, we obtain a factor of $\mathtt{t}^{a-2}$. We repeat this process until we reach the right-most handles of the original portion which requires no action ($\mathtt{t}^0 = 1$). Hence, the number of $\mathtt{t}$ factors needed is

$$c_a = (a-1) + (a-2) + \cdots + 0 = \frac{a(a-1)}{2} \tag{33}$$

The final fork we obtain is $B_{0,a+b}$ which is the standard fork $F_{0,a+b,0}$. Combining all steps we obtain the following equation to decompose $B_{a,b}$.

Lemma 8.3.

$$B_{a,b} = (-\mathtt{q})^a \mathtt{t}^{c_a} F_{0,a+b,0} \tag{34}$$

The decomposition of $A_{a,b,c}$ and $C_{a,b,c}$ are similar. First of all we note that $A_{a,0,c}$ is just the fork $F_{a,c,0}$. Therefore to decompose $A_{a,b,c}$ into standard forks in $H_{n,\ell}$ we just need to distribute b tine edges connecting $(i-1)^{st}$ and i^{th} punctures (the middle porition) to the left and right of the ith puncture. To obtain a generalized formula for distributing b tine edges of the middle portion we need to see what happens when we distribute one tine edge of the middle portion to the left and

right. Suppose we have at least one tine edge in the middle portion of $A_{a,b,c}$. When we distribute one tine edge in the middle to the left and right portions of $A_{a,b,c}$ we will have the sum of two forks one with one more tine edge on the left and one with one more tine edge on the right. However, the tine edge that is distributed to the right portion of the fork must have its handle passed through all a handles on the left and the ith puncture. The decomposition results in the following equation

$$A_{a,b,c} = A_{a+1,b-1,c} + \mathtt{q}\mathtt{t}^a A_{a,b-1,c+1} \tag{35}$$

We can repeat this process with all new forks obtained from the previous decomposition until the middle portion of each fork in the final decomposition is zero. At this point, all forks in the decomposition are standard forks in $H_{n,\ell}$. The general formula of the decomposition is proven by induction and given in the following lemma.

Lemma 8.4.

$$A_{a,b,c} = \sum_{k=0}^{b} \mathtt{q}^k \mathtt{t}^{ak} \binom{b}{k}_{\mathtt{t}} F_{a+b-k,c+k,0} \tag{36}$$

Proof. We prove Equation 36 by inducting on $b \geq 0$. For $b = 0$, the equation reduces to our earlier observation that $A_{a,0,c} = F_{a,c,0}$. We assume that the equation is true for $b-1$. Then, using Equation 35 and the induction hypothesis, we have

$$\begin{aligned} A_{a,b,c} &= A_{a+1,b-1,c} + \mathtt{q}\mathtt{t}^a A_{a,b-1,c+1} \\ &= \sum_{k=0}^{b-1} \mathtt{q}^k \mathtt{t}^{(a+1)k} \binom{b-1}{k}_{\mathtt{t}} F_{a+b-k,c+k,0} + \mathtt{q}\mathtt{t}^a \sum_{k=0}^{b-1} \mathtt{q}^k \mathtt{t}^{ak} \binom{b-1}{k}_{\mathtt{t}} F_{a+b-1-k,c+1+k,0}. \end{aligned}$$

We separate the first term of the first summation and the last term of the second summation then shift the indices of the second summation, recombine the two summations to obtain

$$\begin{aligned} A_{a,b,c} &= F_{a+b,c,0} + \sum_{k=1}^{b-1} \mathtt{q}^k \mathtt{t}^{(a+1)k} \binom{b-1}{k}_{\mathtt{t}} F_{a+b-k,c+k,0} \\ &\quad + \sum_{k=1}^{b-1} \mathtt{q}^k \mathtt{t}^{ak} \binom{b-1}{k-1}_{\mathtt{t}} F_{a+b-k,c+k,0} + \mathtt{q}^b \mathtt{t}^{ab} F_{a,c+b} \\ &= F_{a+b,c,0} + \sum_{k=1}^{b-1} \mathtt{q}^k \mathtt{t}^{ak} \left[t^k \binom{b-1}{k}_{\mathtt{t}} + \binom{b-1}{k-1}_{\mathtt{t}} \right] F_{a+b-k,c+k,0} + \mathtt{q}^b \mathtt{t}^{ab} F_{a,c+b}. \end{aligned}$$

Applying the identities for $\mathtt{t}$-binomials as shown in Proposition 7.3 and collecting the two separated terms into the summation, we obtain the decomposition of $A_{a,b,c}$ into the standard forks in $H_{n,\ell}$

$$A_{a,b,c} = \sum_{k=0}^{b} \mathtt{q}^k \mathtt{t}^{ak} \binom{b}{k}_{\mathtt{t}} F_{a+b-k,c+k,0}. \tag{37}$$

□

The decomposition of $C_{a,b,c}$ follows the same pattern as $A_{a,b,c}$. Therefore following the same derivation we get the following formula

Lemma 8.5.

$$C_{a,b,c} = \sum_{k=0}^{b} \mathtt{t}^{ck} \binom{b}{k}_{\mathtt{t}} F_{0,a+b-k,c+k} \tag{38}$$

The absence of $\mathtt{q}$ in the formula is due to the fact there is no handle wrapping around any puncture in the forks. The shift of the zero entry in the index of the standard fork is due to the shift of the positions of the tine edges between the fork $A_{a,b,c}$ and $C_{a,b,c}$. Now apply Lemma 8.3, 8.4, and 8.5 to the local forks $A_{0,\ell,0}$, $B_{\ell,0}$ and $C_{0,\ell,0}$ we have

Corollary 8.6. *The decompositions of $A_{0,\ell,0}$, $B_{\ell,0}$ and $C_{0,\ell,0}$ into standard local forks in $H_{n,\ell}$ are*

$$A_{0,\ell,0} = \sum_{k=0}^{\ell} \mathtt{q}^k \binom{\ell}{k}_{\mathtt{t}} F_{\ell-k,k,0}. \tag{39}$$

$$B_{\ell,0} = (-\mathtt{q})^{\ell} \mathtt{t}^{c_\ell} F_{0,\ell,0} \tag{40}$$

$$C_{0,\ell,0} = \sum_{k=0}^{\ell} \binom{\ell}{k}_{\mathtt{t}} F_{0,\ell-k,k} \tag{41}$$

8.4 Braid Action under the Specialization $\Phi_\ell(\mathtt{t}) = 0$

Applying Corollary 8.6 to Equation 32 we obtain actions of the braid generators σ_i on $A_{n,\ell}$

Proposition 8.7. *For all $n \geq 2$, $\ell \geq 0$, and $|i-j| > 1$, the action of the braid generators σ_i on the basis of $A_{n,\ell}$ is given by*

$$\sigma_i F_{\boldsymbol{e}_j} = F_{\boldsymbol{e}_j} \tag{42}$$

$$\sigma_i F_{\boldsymbol{e}_{i-1}} = \sum_{k=0}^{\ell} \mathtt{q}^k \binom{\ell}{k}_{\mathtt{t}} F_{\ell-k,k,0} \tag{43}$$

$$\sigma_i F_{\boldsymbol{e}_i} = q^{\ell}(-1)^{\ell} t^{\ell(\ell-1)/2} F_{0,\ell,0} \tag{44}$$

$$\sigma_i F_{\boldsymbol{e}_{i+1}} = \sum_{k=0}^{\ell} \binom{\ell}{k}_{\mathtt{t}} F_{0,\ell-k,k} \tag{45}$$

We note that the unambiguous usage of the notation of local forks of type F in equations in Proposition 8.7 is due to the fact that standard forks in $H_{n,\ell}$ have in total ℓ tine edge connecting the punctures. We also use the definition of c_ℓ as $\frac{\ell(\ell-1)}{2}$.

Proposition 8.7 shows that the action of σ_i on the submodule $A_{n,\ell}$ results in the standard forks of the form $F_{\ell-k,k,0}$ and $F_{0,\ell-k,k}$ for $0 \leq k \leq \ell$, many of which lie outside of the submodule $A_{n,\ell}$. In other words, the braid group action is not closed on $A_{n,\ell}$, and $A_{n,\ell}$ is not a representation of braid group B_n. However under the specialization $\Phi_\ell(\mathtt{t}) = 0$ the coefficients, namely $\binom{\ell}{k}_{\mathtt{t}}$, of these forks in all braid actions vanish for all $0 < k < \ell$ by Lemma 7.4. The only terms that survive in Equations 43, 44 and 45 are the forks in the basis of $A_{n,\ell}$. Therefore, the braid action becomes closed in $A_{n,\ell}$ under the specialization t^ℓ. We summarize the discussion in the following proposition.

Proposition 8.8. *For all $n \geq 2$, $\ell \geq 0$ and under the specialization $\Phi_\ell(\mathtt{t}) = 0$, the braid actions of σ_i are closed in $A_{n,\ell}$. The equations are given by*

$$\sigma_i F_{\boldsymbol{e}_j} = F_{\boldsymbol{e}_j} \tag{46}$$

$$\sigma_i F_{\boldsymbol{e}_{i-1}} = F_{\boldsymbol{e}_{i-1}} + \mathtt{q}^{\ell} F_{\boldsymbol{e}_i} \tag{47}$$

$$\sigma_i F_{\boldsymbol{e}_i} = -\mathtt{q}^{\ell} F_{\boldsymbol{e}_i} \tag{48}$$

$$\sigma_i F_{\boldsymbol{e}_{i+1}} = F_{\boldsymbol{e}_i} + F_{\boldsymbol{e}_{i+1}} \tag{49}$$

The minus sign in Equation 48 can be determined by the consideration of the odd and even cases of ℓ and it can be shown that $(-1)^{\ell} t^{\ell(\ell-1)/2} = -1$ when $t^{\ell} = 1$. We realize that these are the equations for σ_i in the Burau representation $\mathscr{B}_n$ under the identification of parameters $\mathtt{q}^{\ell} \to \mathtt{q}$. This completes the proof of Theorem 8.1.

8.5 Complementary Subrepresentations in the Specialization

The purpose of this section is to prove Theorem 8.2. That is, there exists no complementary subrepresentation to the Burau representation in the specialization $\Phi_\ell(\mathtt{t}) = 0$.

Proof. We prove Theorem 8.2 using proof by contradiction. Suppose there exists a complementary subrepresentation to the Burau representation spanned by forks of the form $F + a$ where $F \in (\mathcal{H}_{n,\ell} - \mathcal{A}_{n,\ell})|_{\Phi_\ell(\mathtt{t})=0}$ and $a \in A_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$. Consider the fork $F + a$ in the complementary subrepresentation where $F = F_{(\ell-1,1,0\ldots,0)}$ and a is a fork in $A_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$. In other words, F is a standard fork in $\mathcal{H}_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$ such that there are $\ell - 1$ tine edges between the first and second puncture and only one tine edge between the second and the third puncture. We show that there exists no a such that the action of σ_1 is closed with respect to the complementary submodule and thus provide a contradiction to the assumption that there is a complementary subrepresentation.

The action of σ_1 on $F + a$ is given by

$$\begin{aligned}\sigma_1(F + a) &= (-\mathtt{q})^{\ell-1}\mathtt{t}^{(\ell-2)(\ell-1)/2}(F_{\mathbf{e}_1} + F) + \sigma_1 a \\ &= \mathtt{q}^{\ell-1}\mathtt{t}(F_{\mathbf{e}_1} + F) + \sigma_1 a \\ &= \mathtt{q}^{\ell-1}\mathtt{t}F + (\sigma_1 a + \mathtt{q}^{\ell-1}\mathtt{t}F_{\mathbf{e}_1})\end{aligned}$$

If there is a complementary subrepresentation to the Burau representation under the specialization $\Phi_\ell(\mathtt{t}) = 0$, the action of σ_1 on $F + a$ must be closed with respect to the submodule. Therefore, we must have

$$\sigma_1(F + a) = \mathtt{q}^{\ell-1}\mathtt{t}(F + a) \tag{50}$$

since $\sigma_1 a + \mathtt{q}^{\ell-1}\mathtt{t}F_{\mathbf{e}_1}$ is contained in $A_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$. Equivalently in $A_{n,\ell}|_{\Phi_\ell(\mathtt{t})=0}$

$$\sigma_1 a = \mathtt{q}^{\ell-1}\mathtt{t}(a - F_{\mathbf{e}_1}) \tag{51}$$

By expressing $a = \sum_j c_j F_{\mathbf{e}_j}$ as a $\mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm 1}]$-linear combinations of basic vectors in $\mathcal{A}_{n,\ell}$, we obtain the following conditions for c_j:

$$\begin{cases} -\mathtt{q}^\ell c_1 + c_2 & = \mathtt{q}^{\ell-1}\mathtt{t}(c_1 - 1) \\ c_j & = \mathtt{q}^{\ell-1}\mathtt{t}c_j \quad \text{for} \quad 2 \le j \le n-1 \end{cases} \tag{52}$$

Therefore, $c_j = 0$ for all j from 2 to $n - 1$. The first equation becomes

$$\mathtt{t} = c_1(\mathtt{t} + \mathtt{q})$$

Since $(\mathtt{t} + \mathtt{q})$ does not have an inverse in the ring of Laurent polynomials $\mathbb{Z}[\mathtt{q}^\pm, \mathtt{t}^\pm]$ with $\mathtt{t}^\ell = 1$, there is no solution for c_1 in the ring of Laurent polynomials $\mathbb{Z}[\mathtt{q}^\pm, \mathtt{t}^\pm]$ with $\mathtt{t}^\ell = 1$. Therefore, there is no complementary subrepresentation to the Burau representation for $n \ge 3$. □

9 Quantum representations of the braid groups via $H_{n,\ell}$

Another interesting representation of the braid groups B_n are the so-called highest weight representation, $W_{n,\ell}$, which are constructed from the quantum algebra $U_q(\mathfrak{sl}_2)$. These are the representations investigated by Jackson and Kerler, and Ekenta and Jackson in [7] and [5]. Without going into too much detail, we briefly introduce these quantum representations following the discussion in [7].

9.1 Quantum Representations

We define a bialgebra $\mathcal{U}$ over $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$ to be generated by the set of elements $\{K^{\pm 1}, E, F^{(n)}\}$ and the following set of relations

$$\begin{aligned} &KK^{-1} = K^{-1}K = 1, \qquad KEK^{-1} = q^2 E, \qquad KF^{(n)}K^{-1} = q^{-2n}F^{(n)}, \\ &F^{(n)}F^{(m)} = \begin{bmatrix} n+m \\ n \end{bmatrix}_q F^{(n+m)}, \qquad \text{and} \qquad [E, F^{(n+1)}] = F^{(n)}(q^{-n}K - q^n K^{-1}). \end{aligned} \tag{53}$$

In addition, the comultiplication $\Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U}$ is defined by

$$\begin{aligned} \Delta(K) = K \otimes K, \qquad \Delta(E) = E \otimes K + \mathbb{1} \otimes E, \\ \Delta(F^{(n)}) = \sum_{j=0}^{n} q^{-j(n-j)} K^{j-n} F^{(j)} \otimes F^{(n-j)}. \end{aligned} \tag{54}$$

To construct a representation of braid groups B_n from $\mathcal{U}$, we consider a representation of $\mathcal{U}$ on a free $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$-module generated by the basic vectors $\{v_0, v_1, \dots\}$, namely a Verma module. The action of the algebra $\mathcal{U}$ on the Verma module V is defined in terms of the action of the generators of $\mathcal{U}$ on the basic vectors v_j:

$$\begin{aligned} K.v_j = sq^{-2j}v_j, \qquad E.v_j = v_{j-1}, \\ F^{(n)}.v_j = \left(\begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{k+j}) \right) v_{j+n}. \end{aligned} \tag{55}$$

This action of $\mathcal{U}$ on V extends to an action of $\mathcal{U}$ on an n-fold tensor product of V via the comultiplication given in Equation 54. In other words, any $x \in \mathcal{U}$ acts on $V^{\otimes n}$ by $\Delta^{(n)}x$, where $\Delta^{(n)} : \mathcal{U} \to \mathcal{U}^{\otimes n}$ is defined recursively by $\Delta^{(2)} = \Delta$ and $\Delta^{(n)} = (\Delta^{(n-1)} \otimes \mathbb{1})\Delta = (\mathbb{1} \otimes \Delta^{(n-1)})\Delta$. It is important for the definition of the spaces $V_{n,\ell}$ and $W_{n,\ell}$ to have the formula for the action of K on $V^{\otimes n}$, that is, the formula for $\Delta^{(n)}(K)$.

$$\Delta^{(n)}(K) = K \otimes \cdots \otimes K \tag{56}$$

A universal R-matrix for $\mathcal{U}$ is given by the following formula

$$R = T \circ C \circ P, \tag{57}$$

where T denotes the usual transposition $T.(v \otimes w) = w \otimes v$. The action of P and C are given by:

$$P.(v_i \otimes v_j) = \sum_{n=0}^{i} q^{\frac{n(n-1)}{2}} \begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{k+j}) v_{i-n} \otimes v_{j+n} \tag{58}$$

$$C.(v_j \otimes v_k) = s^{-(j+k)} q^{2jk} v_j \otimes v_k \tag{59}$$

The universal R-matrix obeys the Yang-Baxter relation given as an equation in $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$ by

$$(R \otimes \mathbb{1})(\mathbb{1} \otimes R)(R \otimes \mathbb{1}) = (\mathbb{1} \otimes R)(R \otimes \mathbb{1})(\mathbb{1} \otimes R). \tag{60}$$

In this context, the Yang-Baxter equation is a braid relation where R corresponds to twisting two strands. Therefore, this allows us to define the action of a braid group generator on $V^{\otimes 2}$ by the following formula:

$$R.(v_i \otimes v_j) = s^{-(i+j)} \sum_{n=0}^{i} q^{2(i-n)(j+n)} q^{\frac{n(n-1)}{2}} \begin{bmatrix} n+j \\ j \end{bmatrix}_q \prod_{k=0}^{n-1} (sq^{-k-j} - s^{-1}q^{k+j}) v_{j+n} \otimes v_{i-n}. \tag{61}$$

We generalize this action of a braid generator on $V^{\otimes 2}$ to the action of σ_i on $V^{\otimes n}$ and obtain a representation of the braid group B_n. Thus, the braid groups B_n can be represented on $V^{\otimes n}$ by the assignment

$$\sigma_i \mapsto \mathbb{1}^{\otimes i-1} \otimes R \otimes \mathbb{1}^{\otimes n-i-1}. \tag{62}$$

The discussion above is summarized in the following theorem in [7]:

Theorem 9.1. *The maps $\sigma_i \mapsto \mathbb{1}^{\otimes i-1} \otimes R \otimes \mathbb{1}^{\otimes n-i-1}$, with R provided in Equation 61, define a representation of the braid group B_n on $V^{\otimes n}$, as a free $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$-module. The maps for σ_i also commute with the action of $\mathcal{U}$ on $V^{\otimes n}$ and preserve the natural $\mathbb{Z}$ grading.*

9.2 The Highest Weight Space $W_{n,\ell}$

To define the highest weight space, let $V_{n,\ell} = \{v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n} \mid (K - s^n q^{-2\ell})(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}) = 0\} \subset V^{\otimes n}$. In other words, $V_{n,\ell}$ is the **kernel** of the action of $K - s^n q^{-2\ell}$ on $V^{\otimes n}$, denoted by $\ker(K - s^n q^{-2\ell})$. The term $s^n q^{-2\ell}$ is the weight of the space. According to the formula for $\Delta^{(n)}(K)$ provided in Equation 56, K acts on $V^{\otimes n}$ in an element-wise manner. By Equation 55, the action of K on $V^{\otimes n}$ can be written as the following

$$\Delta^{(n)}(K)(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}) = s^n q^{-2(\alpha_1 + \ldots \alpha_n)}(v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}) \tag{63}$$

Therefore $V_{n,\ell} = \ker(K - s^n q^{-2\ell})$ is the $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$-span of the vectors $v_{\alpha_1} \otimes \cdots \otimes v_{\alpha_n}$ such that $\alpha_1 + \cdots + \alpha_n = \ell$. We define

$$W_{n,\ell} = \ker(E) \cap V_{n,\ell}. \tag{64}$$

The space $W_{n,\ell}$ is the heighest weight space corresponding to the weight $s^n q^{-2\ell}$. Since the representation of the braid groups B_n on $V^{\otimes n}$ commutes with the action of $\mathcal{U}$ on $V^{\otimes n}$, both $V_{n,\ell}$ and $W_{n,\ell}$ are representations of B_n.

9.3 The Quantum Represention via $H_{n,\ell}$

Jackson and Kerler showed that the quantum representation $W_{n,2}$ is isomorphic to the LKB represetation, $H_{n,2}$ [7]. Ekenta and Jackson extended this result by showing that for any ℓ the representation $\hat{W}_{n,\ell}$ obtained by rescaling the basis of $W_{n,\ell}$ under a diagonal map ε is isomorphic to the Lawrence representation [5]. In general, let $\mathbf{e} = (e_1, \ldots, e_n)$ be a multi-index such that $e_i \in \mathbb{N}$ and $\sum_i e_i = \ell$. These multi-indices index a basis for $V_{n,\ell}$. To prove the previous result, Ekenta and Jackson considered a specific basis $\mathcal{W}_{n,\ell} = \{w_{\mathbf{e}}\}$ of $W_{n,\ell}$ which is rescaled to a basis $\hat{\mathcal{W}}_{n,\ell} = \{\hat{w}_{\mathbf{e}}\}$ of $\hat{W}_{n\ell}$ by the following diagonal map

$$\varepsilon : w_{\mathbf{e}} \to \hat{w}_{\mathbf{e}} = (\mathbf{e})_{q^2}! w_{\mathbf{e}}, \tag{65}$$

where $(\mathbf{e})_{q^2}! = (e_1)_{q^2}! \ldots (e_n)_{q^2}!$. The basis $\hat{\mathcal{W}}_{n,\ell}$ is then mapped to a basis $\mathcal{H}_{n,\ell} = \{F_{\mathbf{e}}\}$ for $H_{n,\ell}$, which is different from the basis $\mathcal{H}_{n,\ell}$ defined in the previous section, via the natural bijective map $F : \hat{w}_{\mathbf{e}} \mapsto F_{\mathbf{e}}$. The map F commutes with the braid action and thus, is a B_n-**equivariant** map. Therefore, F also defines an isomorphism between the two representations $\hat{W}_{n,\ell}$ and $H_{n,\ell}$. The isomorphism between $H_{n,\ell}$ and $\hat{W}_{n,\ell}$ also requires the following identification of parameters

$$\theta : \mathbb{Z}[\mathtt{q}^{\pm 1}, \mathtt{t}^{\pm 1}] \to \mathbb{Z}[s^{\pm 1}, q^{\pm 1}] : \begin{array}{l} \mathtt{q} \mapsto s^2 \\ \mathtt{t} \mapsto -q^{-2} \end{array} \tag{66}$$

In this section, we prove that on the homology side, the representation $H_{n,\ell}$ exhibits a similar behavior. That is, by rescaling the same basis $\mathcal{H}_{n,\ell}$ of $H_{n,\ell}$ by a diagonal map δ, we obtain a representation $\hat{H}_{n,\ell}$ isomorphic to the quantum representation $W_{n,\ell}$. In particular, we define the diagonal map, δ as follows

$$\delta : H_{n,\ell} \to \hat{H}_{n,\ell} : F_{\mathbf{e}} \mapsto \hat{F}_{\mathbf{e}} = \binom{\ell}{\mathbf{e}}_{q^2} F_{\mathbf{e}}, \tag{67}$$

where $\hat{\mathcal{H}}_{n,\ell} = \{\hat{F}_{\mathbf{e}}\}$ is a basis of $\hat{H}_{n,\ell}$ and $\binom{\ell}{\mathbf{e}}_{q^2} = (\ell)_{q^2}!/(\mathbf{e})_{q^2}!$ is the multinomial coefficient. Note that, under the identification of parameters provided in Equation 66, the diagonal map δ is closed under $H_{n,\ell}$. We define $G : \hat{F}_{\mathbf{e}} \mapsto w_{\mathbf{e}}$ as a natural bijective map between $\hat{H}_{n,\ell}$ and $W_{n,\ell}$. Our result is stated as follows

Theorem 9.2. *For any n and ℓ, there is an isomorphism of B_n representations, namely G, over $\mathbb{Z}[s^{\pm 1}, q^{\pm 1}]$*

$$\hat{H}_{n,\ell} \otimes_\theta \mathbb{Z}[s^{\pm 1}, q^{\pm 1}] \quad \xrightarrow{G} \quad W_{n,\ell} \tag{68}$$

that maps the basis of $\hat{H}_{n,\ell}$ to the basis of $W_{n,\ell}$.

The two results are summarized in the following diagram

$$\begin{array}{ccc} \hat{H}_{n,\ell} & \xrightarrow{G} & W_{n,\ell} \\ \delta \uparrow & & \downarrow \varepsilon \\ H_{n,\ell} & \xleftarrow{F} & \hat{W}_{n,\ell} \end{array}, \tag{69}$$

where F and G are B_n-equivariant maps.

Before proving theorem 9.2, we provide an important identity coming from the fact that the map F is B_n-equivariant. Let $\beta \in B_n$ be any braid element. Let $F_{\boldsymbol{\alpha}}, w_{\boldsymbol{\alpha}}, \hat{F}_{\boldsymbol{\alpha}}, \hat{w}_{\boldsymbol{\alpha}}$ be basic vectors. Suppose the action of β on basic vectors $F_{\boldsymbol{\alpha}} \in \mathcal{H}_{n,\ell}$ and $w_{\boldsymbol{\alpha}} \in \mathcal{W}_{n,\ell}$ is given as follows

$$\beta w_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}} w_{\boldsymbol{\gamma}} \qquad \text{and} \qquad \beta F_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}} d_{\boldsymbol{\gamma}} F_{\boldsymbol{\gamma}}, \tag{70}$$

where the sum is on all multi-indices indexing $\mathcal{W}_{n,\ell}$ and $\mathcal{H}_{n,\ell}$. It is true that we have the following identity

Lemma 9.3.

$$(\boldsymbol{\alpha})_{q^2}! c_{\boldsymbol{\gamma}} = (\boldsymbol{\gamma})_{q^2}! d_{\boldsymbol{\gamma}} \tag{71}$$

Proof. We prove Lemma 9.3 by computing the action of β on $\hat{w}_{\boldsymbol{\alpha}}$ in two ways. Using Equation 70, the action of β on $\hat{w}_{\boldsymbol{\alpha}} = (\boldsymbol{\alpha})_{q^2}! w_{\boldsymbol{\alpha}}$ can be computed as follows

$$\beta \hat{w}_{\boldsymbol{\alpha}} = (\boldsymbol{\alpha})_{q^2}! (\beta w_{\boldsymbol{\alpha}}) = \sum_{\boldsymbol{\gamma}} (\boldsymbol{\alpha})_{q^2}! c_{\boldsymbol{\gamma}} w_{\boldsymbol{\gamma}} \tag{72}$$

Since F is an equivariant map, F commutes with the braid action of $\beta \in B_n$. In other words

$$\beta \hat{w}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}} d_{\boldsymbol{\gamma}} \hat{w}_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\gamma}} (\boldsymbol{\gamma})_{q^2}! d_{\boldsymbol{\gamma}} w_{\boldsymbol{\gamma}} \tag{73}$$

By identifying coefficients of Equation 72 and 73, we obtain the identity given in Equation 71. □

In view of Lemma 9.3, the diagonal map ε is defined such that Equation 71 is satisfied and the map F is B_n-equivariant. Similarly, the diagonal map δ is defined in the way that $G : \hat{F}_{\boldsymbol{\alpha}} \mapsto w_{\boldsymbol{\alpha}}$ is B_n-equivariant. The proof is given as follows.

Proof of Theorem 9.2. To show that G is a B_n-equivariant map, we need to show that G commutes with the action of β. That is, to show

$$\beta \hat{F}_{\boldsymbol{\alpha}} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}} \hat{F}_{\boldsymbol{\gamma}} \tag{74}$$

Using Equations 67, 70, and 71 $\hat{F}_{\boldsymbol{\alpha}} = \binom{\ell}{\boldsymbol{\alpha}}_{q^2} F_{\boldsymbol{\alpha}}$, it follows that

$$\beta \hat{F}_{\boldsymbol{\alpha}} = \binom{\ell}{\boldsymbol{\alpha}}_{q^2} \beta F_{\boldsymbol{\alpha}} = \binom{\ell}{\boldsymbol{\alpha}}_{q^2} \sum_{\boldsymbol{\gamma}} d_{\boldsymbol{\gamma}} F_{\boldsymbol{\gamma}} = (\ell)_{q^2}! \sum_{\boldsymbol{\gamma}} \frac{d_{\boldsymbol{\gamma}}}{(\boldsymbol{\alpha})_{q^2}!} F_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}} \binom{\ell}{\boldsymbol{\gamma}}_{q^2} ! F_{\boldsymbol{\gamma}} = \sum_{\boldsymbol{\gamma}} c_{\boldsymbol{\gamma}} \hat{F}_{\boldsymbol{\gamma}}. \tag{75}$$

This completes the proof of Theorem 9.2. □

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