Theory of Energy-Balance Climate Models

GERALD R. NORTH[1]

National Center for Atmospheric Research,[2] *Boulder, Colo. 80303*

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ABSTRACT

A class of mean annual, zonally averaged energy-balance climate models of the Budyko-Sellers type are studied by a spectral (expansion in Legendre polynomials) method. Models with constant thermal diffusion coefficient can be solved exactly. The solution is approached by a rapidly converging sequence with each succeeding approximant taking into account information from ever smaller space and time scales. The first two modes represent a good approximation to the exact solution as well as to the present climate. The two-mode approximation to a number of more general models are shown to be either formally or approximately equivalent to the same truncation in the constant diffusion case. In particular, the transport parameterization used by Budyko is precisely equivalent to the two-mode truncation of thermal diffusion. Details of the dynamics do not influence the first two modes which fortunately seem adequate for the study of global climate change. Estimated ice age temperatures and ice line latitude agree well with the model if the solar constant is reduced by 1.3%.

1. Introduction

Mathematical modeling of climate has received considerable interest in the last few years, because of the suspected delicate equilibrium in which the planet now rests. Schneider and Dickinson (1974) have recently reviewed the progress to date in modeling efforts. These authors delineated a large number of possible factors in global climate change and indicated a hierarchy of models which may be used to incorporate these phenomena.

The simplest models treat only global averages and focus mostly on the radiative transfer properties of the atmosphere under various perturbations. The next members of the hierarchy use zonal and annual averaging but allow latitude dependences of albedo and surface temperature as well as meridional transfer of heat. In these models it is possible to allow the snow or ice line to vary dynamically depending on the climate variables. The ice-albedo feedback mechanism proves to be of greatest importance because of the large contrast in albedo between ice and ice-free areas (Eriksson, 1968). Small changes in the solar constant lead to a change in global temperature which is amplified severalfold by the ice-albedo feedback.

Exploration of models of this type by Budyko (1969) and Sellers (1969) led to several interesting features, among which is the possibility of an abrupt transition to a completely ice-covered earth if the solar constant is lowered by only a few percent. The two approaches differed in complexity yet the results proved to be remarkably similar. The stability properties of these models has been studied numerically by Schneider and Gal-Chen (1973). A model of this same family has also been solved analytically (North, 1975) and the stability properties studied. The results of this study show that three equilibrium climates (Budyko, 1972; Chýlek and Coakley, 1975) obtain with the present value of the solar constant. The present climate as well as an ice-covered earth climate are stable under small perturbations from equilibrium while an intermediate solution with earth about two-thirds covered with ice is unstable and therefore of no physical significance.

Higher members of the hierarchy include global circulation models (GCM) which might even incorporate circulation of the oceans.

The primary focus of this paper will be an analytical study of the class of models first studied by Budyko and Sellers. A fundamental question is how much detail is necessary in the empirical input to guarantee reliable results from the models. This problem is approached by solving a class of such models by spectral methods. The exact solution is approached by a rapidly convergent sequence of approximations with each succeeding approximant incorporating information from successively smaller scales.

The connection between various members of the hierarchy is brought out as it is found that global climate requires only very large space and time scales for an adequate description, while many complications which might be thought to be important are excluded because of scale mismatches.

[1] On sabbatical leave from the University of Missouri-St. Louis.

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2. Model

In order to set up a simple energy-balance climate model it is necessary to assume that all energetic fluxes can be parameterized by the temperature at the earth's surface (sea level). For example, although the characteristic temperature of the earth as a radiating body is determined by the temperature of the atmosphere far above the surface, this temperature is linearly related to that at the surface by the assumption of a constant lapse rate. The empirical formula used in this study is

$$I=A+BT, \tag{1}$$

where I is the outgoing infrared radiation flux (W m^{-2}) T the surface (sea level) temperature (°C), and A = 211.2 W m^{-2}, B=1.55 W m^{-2} $(°C)^{-1}$. The constants A and B are derived from linearizing the Sellers' radiation formula (Sellers, 1969) about 0°C. The linearization is reasonable here because $|T|$ never exceeds 30°C for climates of interest, and in this range it approximates the Sellers' formula to within 1%. The constants A and B are a few percent larger than those suggested by Budyko (1969) and used by North (1975).

The latitude is most easily characterized in terms of the variable x, which is the sine of latitude. The infinitesimal dx is proportional to the area of a strip parallel to the latitude circle x. The formula (1) is assumed to hold for each latitude x so that I and T are considered to be functions of x.

The absorbed solar heating is given by the form $QS(x)a(x,x_s)$, where Q is the solar constant divided by 4, $S(x)$ is the mean annual meridional distribution of solar radiation which is normalized so that its integral from 0 to 1 is unity, and the absorption coefficient $a(x,x_s)$ is one minus the albedo of the earth-atmosphere system. The quantity x_s is the sine of latitude of the ice or snow line. The function $S(x)$, which is determined from astronomical calculations, is uniformly approximated within 2% by

$$S(x)=1+S_2P_2(x), \tag{2}$$

with $S_2=-0.482$ (North, 1975), and $P_2(x)$ is the second Legendre polynomial, $(3x^2-1)/2$. The first term insures the normalization since the second term vanishes upon integration from 0 to 1. The absorption function used is

$$a(x,x_s)=\begin{cases} b_0, & x>x_s \\ a_0+a_2P_2(x), & x<x_s \end{cases} \tag{3}$$

where $b_0=0.38$ is taken from Budyko (1969) as the absorption coefficient over ice or snow 50% covered with clouds, and $a_0=0.697$, $a_2=-0.0779$ come from Fourier-Legendre analyzing the albedo distribution used by Sellers (1969). The latitude-dependent albedo manifested in $a_2P_2(x)$ over ice-free areas takes into account zenith angle dependences as well as the nonhomogeneous mean annual cloud distribution. Budyko (1969) and North (1975) used $a_2=0$ and a smaller value for a_0. Eq. (3) still differs from Sellers' absorption since he allows b_0 to have a temperature dependence. The present model remains soluble even with such a dependence, but the solutions will not be discussed here.

The ice line is determined dynamically in this model by the condition (Budyko, 1969)

$$\left.\begin{array}{ll} T>-10°\text{C}, & \text{no ice present} \\ T<-10°\text{C}, & \text{ice present} \end{array}\right\}. \tag{4}$$

In terms of the function $I(x)$ this condition reads

$$I(x_s)=I_s=195.7 \text{ W m}^{-2}, \tag{5}$$

which follows directly from (1) and (4).

In equilibrium at a given latitude the incoming absorbed, radiant heat is not precisely matched by outgoing radiation—the difference being made up by the meridional divergence of heat flux. In the present study this flux divergence will be modeled by $-\nabla\cdot(D\nabla T)$, where D is a phenomenological thermal diffusion coefficient (Adem, 1962; Fritz, 1960). In terms of the variable x this quantity becomes (in spherical coordinates)

$$-\frac{d}{dx}(1-x^2)D\frac{d}{dx}T(x), \tag{6}$$

where a factor of earth radius squared has been absorbed into D.

In most models it is more convenient to use $I(x)$ as the dependent variable rather than the surface temperature $T(x)$. The two are related by Eq. (1), and in the interpretation of results they may be thought of interchangeably.

The energy balance equation may then be written in a time-dependent form:

$$C\frac{\partial I(x,t)}{\partial t}-\frac{\partial}{\partial x}(1-x^2)D\frac{\partial}{\partial x}I(x,t)+I(x,t) = QS(x)a(x,x_s), \tag{7}$$

where the factor C is the heat capacity of the relevant layers of the atmosphere plus hydrosphere divided by B from Eq. (1), and D has likewise absorbed a factor of B. In Eq. (7), D is dimensionless and C has units of time; it therefore establishes the time scale. Eq. (7), coupled with the condition (5) and simple boundary conditions at $x=0$ and 1, completely specify the problem. Of major interest is the question of how the solutions behave as a function of Q.

It has been shown (North, 1975) that Eq. (7) can be solved analytically in the case of constant D. The results indicated that three climates obtain for the present value of Q: the present climate, an ice-covered earth solution, and an intermediate case with earth more than half-covered with ice. A linear stability analysis showed that the present climate and the ice-covered earth are stable under small perturbations while the intermediate solution is unstable, and, therefore, of no interest. An interesting consequence of the stability analysis was that small departures from the stable solutions decayed back to equilibrium exponentially in time. This relaxation time is characterized by the thermal inertia coefficient C in (7). These results are consistent with the numerical experiments of Schneider and Gal-Chen (1973) as well as the recent GCM results reported by Manabe and Wetherald (1975). For an earth covered with water 75 m deep the characteristic time for decay is about 7.4 years. This depth corresponds roughly to the annual average of the mixed layer thickness (Schneider and Dickinson, 1974).

3. Solution by spectral method

The expansion of latitude-dependent variables in Legendre polynomials was applied to simple climate models as long as 15 years ago (Fritz, 1960). However, the recent studies of climate models seem to have overlooked this powerful technique. The Legendre polynomials have the special advantage that they are the eigenfunctions of the spherical diffusion operator:

$$\frac{d}{dx}(1-x^2)\frac{d}{dx}P_n(x)=-n(n+1)P_n(x). \tag{8}$$

A mean annual model with symmetric hemispheres must satisfy the boundary condition that the gradient of $T(x)$ [or $I(x)$] must vanish at pole and equator. Each even-indexed $P_n(x)$ has zero gradient at the pole and equator, i.e.,

$$(1-x^2)^{\frac{1}{2}}\frac{d}{dx}P_n(x)=0;\quad x=0,\,1, \tag{9}$$

which means that if $I(x)$ [or $T(x)$] is written

$$I(x)=\sum_{n\text{ even}} I_nP_n(x), \tag{10}$$

the expansion will satisfy the boundary conditions term by term. This property is important since the series (10) may be truncated at any order without violating the boundary conditions. Only even terms are required in (10) since $I(x)$ is an even function of x in a mean annual model with symmetry between hemispheres. Odd terms, however, would be necessary in a seasonal model in order to account for a gradient of $I(x)$ at the equator.

To find equilibrium solutions of the model, we set $\partial I/\partial t$ equal to zero, and substitute (10) into (7). After making use of the orthogonality of the P_n we obtain

$$[n(n+1)D+1]I_n=QH_n(x_s), \tag{11}$$

where

$$H_n(x_s)=(2n+1)\int_0^1 S(x)a(x,x_s)P_n(x)dx. \tag{12}$$

The $H_n(x_s)$ [insolation components] may be found analytically from the known forms of $S(x)$ and $a(x,x_s)$; the results are polynomials in x_s. A convenient form for the first few $H_n(x_s)$ will be given in the Appendix.

The first few realizations of (11) are:

$$I_0=QH_0(x_s),\ n=0 \tag{13}$$

$$(6D+1)I_2=QH_2(x_s),\ n=2 \tag{14}$$

$$(20D+1)I_4=QH_4(x_s),\ n=4. \tag{15}$$

Eq. (13) has a simple interpretation; it is the statement that the planetary average temperature is given by the integral of absorption weighted by the meridional distribution of solar flux. The transport property D does not enter (13) since the planetary average of meridional flux divergence must vanish. The second mode amplitude I_2 is roughly a measure of the equator-to-pole temperature difference, and this quantity is influenced by the strength of the meridional heat flux, D. Higher modes are suppressed because of the $n(n+1)$ factor on the left of (11) and the fact that insolation components $H_n(x_s)$ fall off rapidly as a function of n.

If the $C(\partial I/\partial t)$ term is retained in (7), Eq. (11) has an additional term equal to $+C\dot{I}_n$ on the left-hand side (dot refers to time derivative). To get an idea of the time scales for different modes let Q suddenly vanish. The decay time for mode n is $C/[n(n+1)D+1]$. The higher modes are quickly dissipated by the diffusive transport mechanism. Presumably these times also characterize the adjustment to any new value of Q. It is not unexpected that smaller space scales adjust quickly to external changes, while large scales respond slowly.

The ice line is determined by (11) and (5), which may be expressed as

$$I_s=\sum_{n\text{ even}} I_nP_n(x_s). \tag{16}$$

The function of primary interest is the ice edge x_s as a function of Q. Since it is known (Budyko, 1972; Chýlek and Coakley, 1975) that this is a multiple-valued function it is much more convenient to reverse the question and ask for Q as a function of x_s since this is a single-valued function (North, 1975).

To solve the model, given x_s, we divide (11) through by the coefficient of I_n, then multiply each side by $P_n(x_s)$ and sum. The resulting left-hand side is the statement (16). Solving for $Q(x_s)$ we obtain

$$Q(x_s)=I_s\left[\sum_{n\text{ even}}\frac{H_n(x_s)P_n(x_s)}{n(n+1)D+1}\right]^{-1}. \tag{17}$$

Eq. (17) may now be inserted into (11) to calculate any mode amplitude, given x_s. When all modes are retained, Eq. (11) is equivalent to the exact expression given earlier (North, 1975) which involved hypergeometric functions.

Now let us consider the two-mode truncation of the series (10), (16), (17). The present climate may be

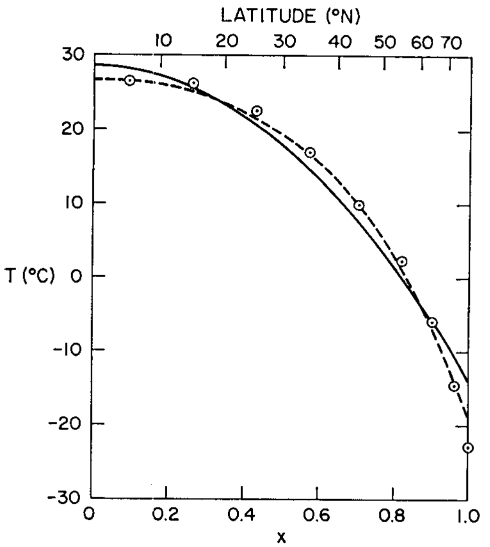

FIG. 1. Sea level temperature as a function of latitude for the present climate. Circles represent observations (Sellers, 1965). The solid line is the temperature calculated from the two-mode approximation. The dashed line illustrates how a third-mode ($n=4$) contribution of proper amplitude (-5°C) would improve the agreement; the latter was *not* claculated from the model.

computed from this system of three equations. Eq. (13) may be used to compute the hemispheric average temperature. Inserting $Q=334.4$ W m^{-2}, $x_s=0.95$, $H_0(0.95)=0.698$, we obtain $I_0=233.4$ W m^{-2} or $T_0=14.3$°C. The second mode amplitude may be computed from the two term version of (16):

$$I_s=I_0+I_2P_2(x_s).$$

The result is $I_2=-44.02$ W m^{-2} which corresponds to $T_2=-28.4$°C. Fig. 1 shows a plot (solid line) of the

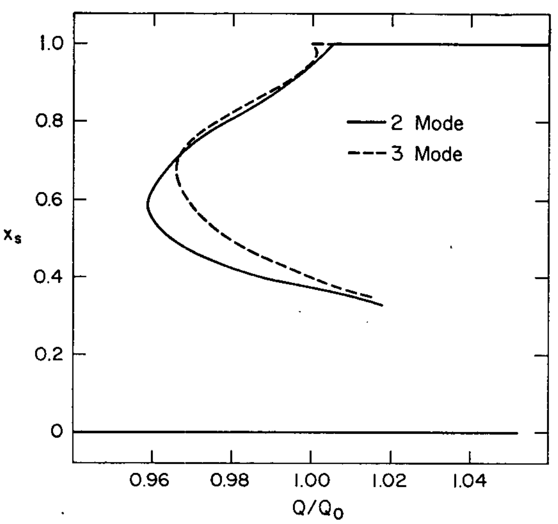

FIG. 2. The sine of latitude of the ice edge, x_s, as a function of the solar constant in units of its present value for the constant D model. The two- and three-mode approximations are shown.

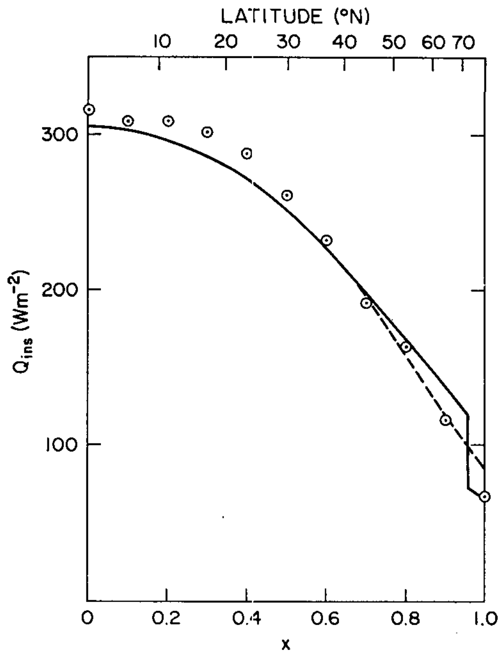

FIG. 3. Radiant energy absorbed by the earth-atmosphere system. Circles represent satellite observations (Raschke *et al.*, 1973) slightly corrected to match the solar constant used in the model rather than that used in data analysis. The solid line represents the heat absorbed as used in the model input. The dashed line is the absorption as sampled in a two-mode approximation.

corresponding temperature distribution vs x. The agreement with observations is surprisingly good considering the crudeness of the model. The heat transport coefficient D may now be computed from (14) since I_2 is now known; the value of $H_2(0.95)$ is -0.434, leading to the result $D=0.382$.

The two-mode truncation of (17) is

$$Q(x_s)=I_s\left[H_0(x_s)+\frac{H_2(x_s)P_2(x_s)}{6D+1}\right]^{-1}. \qquad (18)$$

Fig. 2 shows the comparison of the two-mode function (18) with the three-mode approximation similarly obtained.

The satellite observations of solar heat absorbed by the earth-atmosphere system are shown in Fig. 3. With these data are shown the model input values computed from the right-hand side of (7). Also shown is the sum of the first two Fourier-Legendre components, which are all that contribute to the two-mode solution. This latter should not be considered an approximation to the insolation but rather the only part that plays a role in determining I_0, I_2, $Q(x_s)$, etc., in the two-mode approximation. It is interesting that although the input albedo has a strong discontinuity at x_s, the few-mode projections smooth out this break. The few-mode projection in this case more closely resembles nature than the hypothetical input.

Perhaps the most critical measure of the internal consistency of the model comes from a computation of the total heat flux crossing latitude x. In the two-mode approximation this function $F(x)$ has the simple form

$$F(x)=(6\pi R^2 B)x(1-x^2)T_2 D, \tag{19}$$

where B is defined in (1), R is the radius of the earth, $T_2=-28.4°C$, and $D=0.382$ as computed earlier. Fig. 4 shows the results (solid line) compared with the observations (Sellers, 1965). $F(x)$ is very sensitive to the value of D computed. In the case of the Budyko albedo and radiation formula, D is too small by about 25% (North, 1975). The reason for the extreme sensitivity is that the meridional flux divergence is the difference between much larger quantities representing incoming and outgoing radiant energy. The maximum of $F(x)$ is only about 5% of the energy insolated by the hemisphere (Sellers, 1965).

The dashed line in Fig. 1 shows a three-mode fit to the present temperature distribution. The value of T_4 is *assumed* to be $-5°C$; T_0 and T_2 are taken from the two-mode solution. Clearly a third-mode contribution has the right form to approximate better the present climate. If the constant D model of this section is used to calculate T_4, the value has the right sign but is only about $-0.5°C$. Also the third-mode contribution distorts the $F(x)$ curve (Fig. 4) toward a better fit but the improvement is not significant. The source of error in the third mode comes from the low quality of input in the albedo and radiation formula as well as the diffusion hypothesis itself. The way in which the details of the

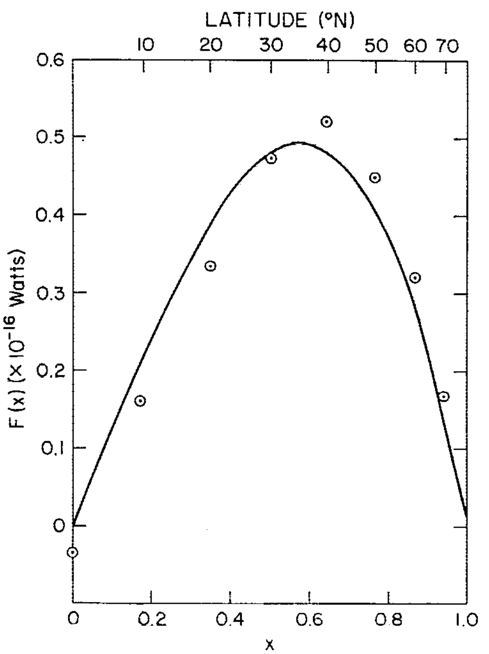

FIG. 4. Total heat flux crossing latitude circles. Circles represent observations (Sellers, 1965) and the solid line is computed from the two mode approximation [Eq. (19)].

TABLE 1. Values of ice edge (x_s) versus derived, necessary solar constant in units of the present solar constant Q_0 (1338 W m^{-2} or 1.92 cal cm^{-2} min^{-1}) for a constant D model. Also shown are the hemispheric average temperature T_0. The subscript outside the parentheses denotes the number of modes retained. Corresponding values of T_2 may be computed from the relation $T_2=-(T_0+10)/P_2(x_s)$.

x_s	$(Q/Q_0)_2$	$(T_0)_2$	$(Q/Q_0)_3$	$(T_0)_3$
0.60	0.959	−8.54	0.969	−7.20
0.65	0.961	−5.15	0.967	−4.35
0.70	0.966	−1.62	0.967	−1.44
0.75	0.977	1.95	0.970	1.62
0.80	0.980	5.43	0.976	4.89
0.85	0.987	8.71	0.985	8.31
0.90	0.994	11.70	0.994	11.64
0.95	1.000	14.32	1.000	14.34
1.00	1.005	16.57	0.999	15.53

dynamics first emerge in the third node will be elaborated in the following sections. Fortunately, two modes give a very good estimate of the present climate and may be all that is necessary for the study of global climate change.

A simple characteristic predicted by climate models is the sensitivity coefficient $Q(dt_0/dQ)$, which gives the global average temperature change ΔT_0 resulting from an infinitesimal fractional change in solar constant, $\Delta Q/Q$. This quantity can be estimated from Table 1 to be about 400°C for the two-mode approximation, corresponding to a 4°C drop in T_0 if Q is lowered by 1%. Retention of a third mode changes this number on the order of 10–20%. The difference should not be considered significant, but rather a rough measure of the uncertainty. It is significant, however, that the sensitivity coefficient is only 140°C without ice-albedo feedback. Amplification by ice-albedo feedback disappears as x_s approaches 1.

4. Budyko transport and the spectral solution

One objective of the present study is to understand the relationship among various members of the model hierarchy. The transport parameterization of Budyko (1969) comes from an analysis of empirical data. A form of the mean annual meridional heat flux divergence which matched observations was found to be

$$\beta[T(x)-T_p], \tag{20}$$

where $T(x)$ is the mean temperature, at latitude whose sine is x, T_p is the hemispheric average temperature, and β is an empirical constant found by Budyko from a scatter diagram.

We now consider the diffusive heat divergence operator

$$-D\frac{d}{dx}(1-x^2)\frac{d}{dx}$$

applied to a two-mode temperature distribution,

$T(x)=T_0+T_2P_2(x)$; there results

$$6DT_2P_2(x)=6D[T(x)-T_0]. \tag{21}$$

Identifying $6D$ with β and noting that $T_0=T_p$, it follows that the two-mode solution to a constant D model treats the transport of heat in the same way as does Budyko. Using Budyko's value $\beta=0.235$ K cal cm^{-2} month^{-1} per degree latitude, we may compute the corresponding dimensionless D used here. The result from converting the units is $D=0.402$ to be compared with the value 0.382 found self-consistently in the last section.

Budyko (1974) comments that while (20) works well on a mean annual basis the value of β must have a seasonal dependence to account for heat advection for individual seasons. The above analysis can shed light on this problem if the advection is actually diffusive. The reason for the failure under this assumption is due to the presence of odd index modes in the individual seasonal temperature distributions. For example, if $T_1P_1(x)$ is present in $T(x)$, Eq. (20) no longer follows from a diffusion model. The presence of $T_1P_1(x)$ also accounts for the failure of the form (20) to match the data in the tropics for individual seasons (Budyko, 1974). D may also have a dependence on T_2 which varies enormously from one season to the next, but is of little significance to climate change in a mean annual treatment (to be shown in a later section).

5. Possible latitude dependence of D

Latitude dependence of ocean boundaries, mountains, persistent zones of baroclinic instability, and large-scale Hadley circulations indicate that D might require an x dependence, say $D=D_0f(x)$. First consider a two-mode approximation by inserting the usual truncated series (10) into the time-independent form of (7). After multiplying by $(2n+1)\,P_n(x)$ and integrating from 0 to 1, we find that Eq. (13) is unaltered.

Eq. (14) is also unaltered except that D in that expression is replaced by D' which may be computed from

$$D'=-5D_0\int_0^1 P_2(x)\frac{d}{dx}\left[(1-x^2)f(x)\frac{d}{dx}P_2(x)\right]dx. \tag{22}$$

Now since the integral in (22) is a constant number calculated once and for all, it may be absorbed into the phenomenological constant D_0. The parameter D_0 is to be determined self-consistently as before so that the case $D=D_0f(x)$ is formally equivalent to the constant D case in two-mode approximation.

Any x dependence of D will make a difference if an $n=4$ mode is retained in the analysis. Such a form will couple the $n=2$ and $n=4$ modes, but as we have seen it does not couple the $n=0$ and $n=2$ modes.

The agreement of the constant D model with observations shown in Fig. 4 indicates that D probably is constant as a function of x, at least on a mean annual basis, since any x dependence would modulate the $x(1-x^2)$ form of the curve and weaken the agreement. If D were given the form $D_0+D_2P_2(x)$, we could possibly reproduce the value $T_4=-5°$C mentioned in the last section by adjusting the ratio D_0/D_2. The addition of another adjustable parameter does not seem justified considering the other uncertainties in the model.

6. Inclusion of systematic circulation effects

In the tropics the constant diffusion hypothesis does not hold for the individual agents of transport in the real atmosphere. For example, atmospheric sensible heat is actually carried toward the equator in very low latitudes. Such large-scale effects might be modeled by including a heat energy flow term on the left of (7). Following Sellers (1969) we might add a term of the form $V(x)(1-x^2)^{\frac{1}{2}}(d/dx)I(x)$ to simulate this type of heat flux divergence. The dimensionless meridional velocity $V(x)$ is to be determined empirically by fitting the transport of sensible heat by the atmosphere.

We now consider the two-mode approximation to the new system. If the velocity field $V(x)$ is divergenceless, Eq. (13), for the $n=0$ mode, is unchanged. The formal interchange of $\mathbf{V}$ and ∇ must be done before zonal and vertical averaging process, since the condition of divergence-free $V(x)$ is too constraining in one dimension. Multiplying the new (time-independent) form of (7) by $(n+1)P_n(x)$ and integrating from 0 to 1, we obtain in place of (14):

$$(6D+1)I_2+I_25\int_0^1 V(x)P_2(x)(1-x^2)^{\frac{1}{2}}(3x)dx=QH_2(x_s).$$

No matter what the functional form of $V(x)$ is, the integral is a constant and it may be absorbed by the phenomenological coefficient D.

The argument above can be readily extended to the case of latent heat transport and oceanic heat transport.

The conclusion is that any organized (non-turbulent) transport by a divergence-free velocity field cannot be distinguished from diffusion in the two-mode approximation to a zonally averaged model.

7. Nonlinear diffusion of heat

Since atmospheric meridional circulation is presumably driven by the temperature differences from one latitude to another, it is not unreasonable to suppose that D might be linearly related to the local gradient of $T(x)$. Stone (1973) presents arguments based on seasonal data that this might be the case. In two-mode analysis it may be shown (as in the previous sections) that the local gradient may be replaced by $[T(1)-T(0)]$, the equator-to-pole temperature difference. In the two-mode problem this is equivalent to the replacement

$$D=K+K_2I_2. \tag{23}$$

When (23) is inserted into (14) we see that only the sum D is determined by fitting the present climate. The relative weights of K and K_2 may be chosen from other empirical considerations. It is interesting that in the two-mode analysis the fit to present climate (Fig. 1 and Fig. 4) is the same as the constant D case independent of the ratio K/K_2. Once this ratio is chosen we obtain for Eq. (14):

$$(6K+1)I_2+6K_2I_2^2=QH_2(x_s). \qquad (24)$$

Eqs. (13) and (18) may be used to eliminate I_2 in (24); the result is a quadratic equation for Q:

$$\alpha Q^2+\beta Q+\gamma=0, \qquad (25)$$

where

$$\left.\begin{aligned}\alpha&=6K_2H_0^2\\ \beta&=-(6K+1)H_0P_2-12K_2H_0I_s-H_2P_2\\ \gamma&=(6K+1)I_sP_2+6K_2I_s^2\end{aligned}\right\},$$

and the functions H_0, H_2, P_2 are to be evaluated at x_s. Eq. (25) admits two roots, one of which may always be rejected on physical grounds, since it has the poles warmer than the equator ($T_2>0$) while the model automatically (and incorrectly here) places the ice albedo north of x_s. The physical branch of the $Q(x_s)$ curve is remarkably close to the constant D case illustrated in Fig. 2. The special but extreme case of $D=K_2T_2$ with K_2 self-consistently determined to be -0.0130 is illustrated in Table 2. These values are to be compared with the linear diffusion case illustrated in Table 1. Though it is not shown here, a similar insensitivity appears when D takes the form $K+K_0I_0$.

The explanation of the insensitivity to nonlinear diffusion mechanisms can be constructed as follows. The formula (18) is a correct expression even if D is an arbitrary function of I_0, I_2, or x_s. It is only an implicit algebraic expression, however, since D now depends on Q through D's dependence on I_0, I_2. The only constraint on D is that it assumes the value 0.382 at $x_s=0.95$, the present climate. As x_s is lowered from 0.95 the second term in parentheses of (18) diminishes because of the factor $P_2(x_s)$. In fact, this term vanishes at the zero of $P_2(x_s)$ which occurs at $x_s=3^{-\frac{1}{2}}\approx0.577$. Hence $Q(x_s)$ becomes independent of D at $x_s=0.577$. This value of x_s is very near the transition to ice-covered earth in all the models studied thus far. This universal value of Q is given by

$$Q(3^{-\frac{1}{2}})=I_s/H_0(3^{-\frac{1}{2}}), \qquad (26)$$

independent of the form or value of D. Now since $Q(0.95)$ is fixed by the present climate and $Q(3^{-\frac{1}{2}})$ is fixed by (26) we can conclude that the function $Q(x_s)$ is qualitatively unaffected by the details of the transport mechanism.

The critical value of I_0 or planetary average temperature at $x_s=3^{-\frac{1}{2}}$ can be computed from (13) by inserting (26); the result is that $I_0=I_s$ or $T_0=-10°C$. In other words, the transition to an ice-covered earth occurs when the planetary *average* temperature is lowered to approximately the temperature which determines the snow line.

TABLE 2. As in Table 1 except D has the nonlinear form $-0.0135T_2$. These results are for the two-mode approximation.

x_s	Q/Q_0	T_0
0.60	0.957	−8.76
0.65	0.958	−5.51
0.70	0.962	−2.12
0.75	0.968	1.35
0.80	0.975	4.76
0.85	0.985	8.42
0.90	0.992	11.40
0.95	1.000	14.32
1.00	1.016	18.23

Examination of the quadratic (25) shows that it also gives the value (26), when $x_s=3^{-\frac{1}{2}}$.

It is also possible to treat the cases described above in a three-mode analysis. The resulting equation for $Q(x_s)$ becomes a cubic [analogue to (25)]. In this case two of the three roots are unphysical and the third closely resembles the two-mode case. The corrections to two-mode results are again small and about the size of those in Table 1 for the constant D case. The nonlinear effects do not introduce significant distortions through the coupling of higher modes.

It is not difficult to carry out a linear stability analysis about the equilibrium solutions using the time-dependent Eq. (7). The method has been illustrated previously (North, 1975) and will not be repeated here. It suffices to say that in two-mode analysis the nonlinear effects do not affect the stability rules already established, i.e., that the present climate as well as the ice-covered earth solution are stable, while the intermediate branch ($0<x_s\lesssim3^{-\frac{1}{2}}$) is unstable under small perturbations away from equilibrium.

While this analysis shows that nonlinear diffusion is not important in long-term global climate change, it is likely to be important in seasonal models. To see this contrast, compare the change in equator-to-pole temperature difference from winter to summer (58°C to 38°C) with the changes in mean annual values from the present to ice age conditions (43°C to 33°C).

8. Transport suppressed by ice cover

Another form of the diffusion coefficient which might have an important effect on climate change is the possibility that heat advection by the oceans is suppressed when they are covered with ice. A simple form for the diffusion coefficient under these circumstances is

$$D=D_0+D_1h(x,x_s), \qquad (27)$$

where $h(x,x_s)$ is a step function having the value 1 for $x<x_s$ and 0 for $x>x_s$. While this model may be solved exactly as indicated by North (1975), we proceed in the spirit of the last few sections of this paper with a two-

TABLE 3. As in Table 1 except the model has a step function D suppressing transport north of x_s by the fraction denoted by the subscript outside the parentheses. All results shown are for two-mode approximation.

x_s	$(Q/Q_0)_{0.3}$	$(T_0)_{0.3}$	$(Q/Q_0)_{1.0}$	$(T_0)_{1.0}$
0.60	0.960	−8.32	0.966	−7.56
0.65	0.966	−4.49	0.980	−2.58
0.70	0.972	−0.74	0.990	1.76
0.75	0.979	2.88	0.996	5.30
0.80	0.985	6.23	0.999	8.22
0.85	0.991	9.13	1.000	10.58
0.90	0.996	11.98	1.000	12.56
0.95	1.000	14.32	1.000	14.32
1.00	1.004	16.37	1.002	16.14

mode analysis. A straightforward analysis shows that (13), (14) and (18) are still valid except that D is replaced by $J(x_s)$, which is given by

$$J(x_s)=3D_1(x_s^3-\tfrac{3}{5}x_s^5)+D_0. \qquad (28)$$

As in the previous section D_0 and D_1 are constrained by the fit to the present climate, i.e., $J(0.95)=0.382$. Using this constraint plus the demand that D_1 be about 30% of (D_0+D_1) in accordance with the fraction of energy carried by oceans, we obtain $D_0=0.250$ and $D_1=0.107$. Table 3 shows the resulting $Q(x_s)$, $T_0(x_s)$, and $T_2(x_s)$ values for this model. The argument from the preceding section explains the insensitivity to this nonlinear effect. As an extreme illustration the case $D_0=0$ (no heat transport over ice) is also shown in Table 3. This case is rather different from the others since the curvature of the $Q(x_s)$ function is different near $x_s=0.95$. This extreme case is probably not well approximated by the two-mode analysis since the temperature is surely other than quadratic in x near the ice edge.

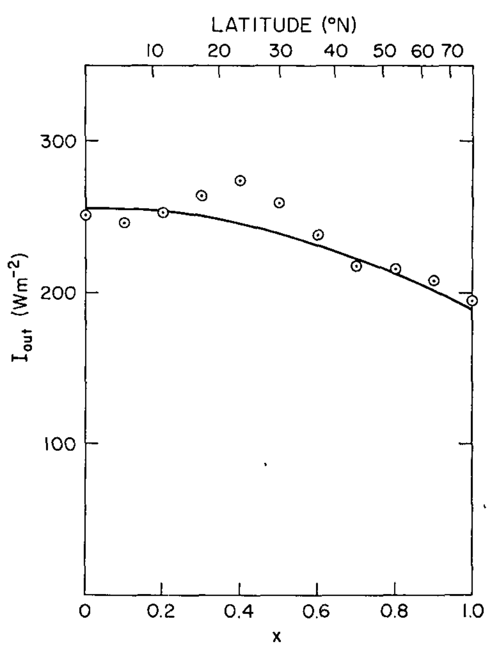

FIG. 5. Outgoing infrared energy flux versus latitude. Circles represent satellite data (Raschke *et al.*, 1973), and the solid line is computed from the two-mode approximation. Discrepancies are attributed to the radiation formula (1); see text.

9. Effects of nonhomogeneous cloudiness

Fig. 5 shows the computed outgoing radiation flux compared with satellite data (Raschke *et al.*, 1973). The formula (1) cannot possibly reproduce the finestructure of the data since the observed $T(x)$ is nearly quadratic in x. The discrepancy must lie in the effects of nonhomogeneous cloudiness, lapse rate, absolute humidity, etc. The present study seems to focus the difficulty directly on the radiation formula rather than on transport phenomena. This section will resolve the discrepancy of Fig. 5 and show that the results of the previous sections are unaffected by the necessary modifications.

If nonhomogeneities in lapse rate or opacity to infrared radiation are responsible for the difficulties shown in Fig. (5), we may modify (1) by allowing A and B to be functions of x. Such an approach is consistent with both the Budyko (1969) and Sellers (1969) approaches. It is also consistent with a formula suggested by Manabe and Wetherald (1967) based on a model atmosphere calculation. If observed cloudiness (Sellers, 1965) is inserted into any of the three empirical formulas the satellite results are not well reproduced. However, our approach will be to allow A to be a function of x, and to find this function directly from the satellite data.

The function $A(x)$ may be found most easily by a Fourier-Lengendre analysis of the difference of the two curves in Fig. 5:

$$A(x)=\sum_{n\ \mathrm{even}} A_nP_n(x), \qquad (28)$$

where A_0 is the constant value used in the earlier sections of this paper and A_2 is found to be 0.84 W m^{-2}. The inclusion of only the first two terms of (28) does not change the quadratic shape for the function $I(x)$, so that many higher order terms are required to reproduce the structure of the data. It is interesting to note, however, that when the two-mode approxmation is applied to the energy balance equation, only the A_0 and A_2 amplitudes are coupled to the solutions I_0 and I_2. All higher order terms, no matter how large, are not coupled to the two-mode temperature amplitudes. The situation is analogous to the case of an x dependent diffusion coefficient studied earlier. Though the details will not be given here, the new formula for $Q(x_s)$ in the two-mode approximation when all terms of (28) are retained is

$$Q(x_s)=Q_0(x_s)\left[1+\frac{A_2P_2(x_s)}{I_s(6D+1)}\right], \qquad (29)$$

where $Q_0(x_s)$ is given by (18). The second term in the brackets of (29) has a value of about 0.001 for the present climate, and becomes smaller as x_s decreases. The conclusion is that the outgoing radiation formula may be modified to fit the data to any desired accuracy without having a significant effect on any of the two-mode calculations in this paper.

It is likely that the constant B should also be a function of x, though this cannot be uniquely extracted from Fig. 5. In the event that such dependence can be extracted from future satellite data (for example, of seasonal variation), it will be necessary to include both the Fourier-Legendre components B_2 and B_4 in the two-mode approximation.

10. Ice age conditions

The last major advance of the ice sheet corresponded to a mean annual ice-covered area of about three times the present mean annual value. In terms of the model the ice age called for $x_s=0.85$. If we insert these values into (18), using $H_0(0.85)=0.680$, $H_2(0.85)=-0.495$, we obtain $Q(0.85)=331.0$ W m^{-2} which is 1.3% lower than the assumed present value. The calculated planetary average temperature, $T_0=8.7$°C, is 5.6°C below the present value. The temperature at the equator is lowered by 3.8°C while at the pole the reduction is 9.3°C. These figures seem to be in good agreement with the estimates of paleotemperatures given by Flint (1971) of 6, 3 and 20°C, respectively. The large discrepancy at the pole may be due to the coupling of higher modes, or the suppression of oceanic heat transport across ice-covered areas. From Fig. 1 it is clear that the largest error incurred in a two-mode truncation is at the pole; convergence is poorest at the pole because $P_n(x)$ assumes its maximum value (unity) at $x=1$.

Though the agreement of the constant D model coupled with an effective decrease of Q by 1.3% is very good, it could hardly be considered a unique explanation at this time. It may be, for example, that similar results can be obtained from a study of natural fluctuations of the climate at constant Q. Such self-induced fluctuations might require a number of non-equilibrium feedback effects not necessary in the quasi-equilibrium treatment of the earlier sections of this paper. For example, the thermal inertia coefficient is likely to be a function of surface temperature, since the thermally responding layers of the oceans would probably thicken as the temperature is lowered. Also, the ice line x_s is not likely to be determined by the instantaneous temperature distribution, but rather to be lagged because of melting time constants, etc.

11. Conclusion

A simple model in the spirit of earlier systems proposed by Budyko and Sellers has been presented and solved analytically by a spectral method. A sequence of approximations is found in which each succeeding correction incorporates ever smaller space and time scale effects. The mechanism for heat transport is based upon a turbulent diffusion approach which is formally equivalent to that of Budyko in the two-mode approximation. After the transport coefficient is determined self-consistently from the present temperature distribution, the model predicts very accurately the total heat transport across latitude circles.

The model has proved to be remarkably insensitive to various generalized forms of the diffusion coefficient such as latitude, temperature, temperature gradient, and ice line dependences. In fact, nonhomogeneous features incorporated in the outgoing radiation formula also appear to have an insignificant affect.

Even though the fit with the present climate is good for the two-mode model, considerable uncertainty exists even in the two-mode input data. The satellite data have not had enough sampling time, and certainly the coefficients in the radiation formula (1) are not known well enough to take even these results literally, Even the solar constant to be used is not agreed upon by the various climate modelers. The values of solar constant (Q), radiation parameters (A, B), and albedo parameters (a_0, a_2, b_0) cannot be changed independently without changing the planetary average temperature (lowest mode output). Most of these parameters are not actually known to better than 5%.

Achieving accuracy at the third-mode level must involve a multitude of hypotheses which are not necessary in a two-mode analysis. Only one or two adjustable parameters in such hypotheses (right or wrong) would be necessary to bring the T_4 amplitude to its observed value while simultaneously improving the fit to the energy transport $F(x)$. The obvious non-uniqueness of such a procedure, however, implies that it may be premature at this time.

Fortunately, in the solvable few-mode models we have studied, the third mode has little influence on climate change. Therefore, the two-mode model provides a useful framework for incorporating updated empirical information in future modeling efforts. The two-mode scheme should also be useful in describing the gross effects of cloud feedback mechanisms once these inputs are more clearly understood.

The few-mode models discussed in this paper seem capable of describing the present climate and ice age conditions with surprising accuracy. It is interesting to speculate, after the fact, as to why such a crude scheme works so well. One possibility is the following. We notice that global climate description on a mean annual basis requires only two modes (meridional length scales of the order of 12,000 km) and the corresponding time scales are of the order of one to two months. On the other hand, the atmospheric fluctuations responsible for heat transport have meridional length scales of the order of the amplitude of the Rossby wave scale (1000 km) and autocorrelation times of the order of three days. These two coupled systems are then grossly mismatched in both length and time scales. In this case the heat transport may be well approximated by a random walk process for purposes of studying global climate change. Use of a diffusion coefficient is known to correctly model the ensemble averages of quantities under the influence of a random walk transport agent.

The above argument has least credibility when applied to oceanic transport. In this case the transport is more organized and may resemble diffusion only in the first few modes.

After this paper was written an interesting study of Budyko-type models by Held and Suarez (1974) appeared. These authors pointed out that the meridional temperature distribution in Budyko's model has a discontinuity at the ice line. This peculiar behavior is naturally removed by the spectral method employed in the present paper. This difficulty in Budyko's original treatment may now be traced to the fact that only two modes are effectively retained in the horizontal flux divergence term while all modes are retained in the insolated heat term of the energy-balance equation. Held and Suarez also noticed that constant D models can be solved analytically. In addition, these authors examined numerically a case with D proportional to temperature gradient with results agreeing with the present conclusions drawn from the two-mode approximation. The lack of sensitivity to such nonlinear effects is also supported by recent numerical experiments performed by Gal-Chen and Schneider (1975).

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APPENDIX

Insolation Components

One advantage of the spectral method is that virtually all computations can be done by hand if only a few modes are retained. The computations are simplified if some of the functions used are given in tabular form. The formulas and tables in this appendix should make it easy to test the effects of alternative parameterizations or hypotheses through the third-mode approximation.

TABLE A1. Values of the incomplete integrals over Legendre polynomials. $G_{ijk}(x_s)$ is defined in the Appendix.

x_s	G_{002}	G_{022}	G_{004}	G_{024}	G_{222}	G_{224}
0.60	−0.1920	0.0770	0.0230	−0.0323	−0.0330	0.0169
0.65	−0.1877	0.0774	0.0020	−0.0341	−0.0330	0.0167
0.70	−0.1785	0.0791	−0.0192	−0.0380	−0.0326	0.0160
0.75	−0.1641	0.0834	−0.0385	−0.0435	−0.0314	0.0144
0.80	−0.1440	0.0915	−0.0533	−0.0494	−0.0281	0.0120
0.85	−0.1179	0.1051	−0.0607	−0.0531	−0.0209	0.0101
0.90	−0.0855	0.1262	−0.0571	−0.0507	−0.0071	0.0118
0.95	−0.0463	0.1570	−0.0384	−0.0358	+0.0172	0.0236
1.00	0.0	0.2000	0.0	0.0	0.0571	0.0571

TABLE A2. Values of functions required for hand computations of $Q(x_s)$, T_0, T_2, T_4 in all models retaining up to three modes. Albedo parameters used are $a_0=0.697$, $a_2=-0.0779$, $b_0=0.380$.

x_s	H_0	H_2	H_2P_2	H_4	H_4P_4
0.60	0.6174	−0.5825	−0.0233	0.1384	−0.0565
0.65	0.6323	−0.5761	−0.0771	0.0821	−0.0352
0.70	0.6461	−0.5635	−0.1324	0.0295	−0.0121
0.75	0.6587	−0.5452	−0.1874	−0.0145	−0.0051
0.80	0.6703	−0.5222	−0.2402	−0.0344	0.0106
0.85	0.6806	−0.4952	−0.2891	−0.0593	0.0030
0.90	0.6898	−0.4656	−0.3329	−0.0536	−0.0112
0.95	0.6977	−0.4345	−0.3709	−0.0272	−0.0151
1.00	0.7045	−0.4031	−0.4031	0.0193	+0.0193

The heating components $H_n(x_s)$ may be written explicitly in terms of the albedo parameters a_0, a_2, b_0 as follows:

$$H_0(x_s)=(a_0-b_0)[x_s+S_2G_{002}(x_s)] +a_2[G_{002}(x_s)+S_2G_{002}(x_s)]+b_0,$$

$$H_2(x_s)=5(a_0-b_0)[G_{002}(x_s)+S_2G_{022}(x_s)] +5a_2[G_{022}(x_s)+S_2G_{222}(x_s)]+S_2b_0,$$

$$H_4(x_s)=9(a_0-b_0)[G_{004}(x_s)+S_2G_{024}(x_s)] +9a_2[G_{024}(x_s)+S_2G_{224}(x_s)],$$

where the $G_{ijk}(x_s)$ are given by

$$G_{ijk}(x_s)=\int_0^{x_s} P_i(x)P_j(x)P_k(x)dx.$$

Values of $G_{ijk}(x_s)$ are given in Table A1.

Table A2 gives values of functions which are useful in making hand computations of more general models.

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